JHEP HYPER VERSION

Received: *November 7, 2000,* Accepted: *January 15, 2001*

Phases of supersymmetric D-branes on Kähler manifolds and the McKay correspondence

Peter Mayr

CERN Theory Division
CH-1211 Geneva 23, Switzerland
E-mail: Peter.Mayr@cern.ch

Abstract: We study the topological zero mode sector of type-II strings on a Kähler manifold X in the presence of boundaries. We construct two finite bases, in a sense bosonic and fermionic, that generate the topological sector of the Hilbert space with boundaries. The fermionic basis localizes on compact submanifolds in X. A variation of the FI terms interpolates between the description of these ground states in terms of the ring of chiral fields at the boundary at small volume and helices of exceptional sheaves at large volume, respectively. The identification of the bosonic/fermionic basis with the dual bases for the non-compact/compact K-theory group on X gives a natural explanation of the McKay correspondence in terms of a linear sigma model and suggests a simple generalization of McKay to singular resolutions. The construction provides also a very effective way to describe D-brane states on generic, compact Calabi-Yau manifolds and allows to recover detailed information on the moduli space, such as monodromies and analytic continuation matrices, from the group theoretical data of a simple orbifold.

Keywords: D-branes, Brane Dynamics in Gauge Theories, Supersymmetric Effective Theories.

JHEP01(2001)018

Contents

---

1. Introduction

The fact that open strings constitute an important sector of type-II closed string theories has not been appreciated appropriately before the seminal works on string duality [1], although it appeared earlier, somehow in disguise, in the mathematical literature on mirror symmetry [2]. As the Dirichlet boundary conditions of the open strings break half of the supersymmetry of the closed string sector, this extends the beautiful geometric structure of the vacua of non-perturbative $\mathcal{N} = 2$ supersymmetric theories [3] described by closed strings, to $\mathcal{N} = 1$ supersymmetry theories in terms of open strings.

It is therefore clearly very important to study this sector of type-II strings which keeps new aspects of mirror symmetry [2, 4, 5] and non-perturbative $\mathcal{N} = 1$ physics. The two-dimensional perspective has been emphasized in the works [6]–[8] and there has been much conceptional progress since then [9]–[14], see [15] for a review and a more complete list of references. In a first step one would like to understand the zero mode structure, that is the open string ground states which represent BPS D-branes. Here we will show that Witten's gauged linear sigma model [16, 17] provides the natural language to construct a finite basis, in a sense to be made precise, for all D-branes on a Kähler manifold X. We follow closely the discussion of [16] which in particular allows to interpolate between D-branes at large and small volume. In the closed string case, the topological Hilbert space $\mathcal{H}^{\text{top}}_{\text{cl}}$ has a large volume interpretation in terms of a deformation of the cohomology ring of X [18]–[20], and a small volume interpretation in terms of the ring of chiral fields of a $(2,2)$ LG theory.[1] We find a very similar structure in the topological sector with boundaries, with the large volume phase described by helices of exceptional sheaves, and the small volume phase by the ring of zero modes of the chiral fields at the boundary.

A key point is that at the boundary, the chiral fields split into representations of the unbroken supersymmetry with bosonic and fermionic statistics in the directions transverse and normal to the brane, respectively.[2] Thus the ordinary ring of zero modes of chiral super-fields splits into a ring of bosons and a ring of fermions. Multiplication of the trivial ground state with the bosonic zero modes leads to a basis $\{R_a\}$ of generators for the topological sector with boundaries $\mathcal{H}^{\rm top}_{\rm op}$ that correspond to an exceptional collection of line bundles which spread over the total space X. Multiplication with fermionic zero modes generates a dual basis $\{S^a\}$ of generators for $\mathcal{H}^{\rm top}_{\rm op}$ that represent sheaves localized on sub-manifolds of X. The latter will be identified with the so-called fractional brane states [9, 12]. We will study in detail a non-trivial example that demonstrates this appealing correspondence between the localization of two-dimensional fermions and that of the associated fractional branes.

The multiplication rule of the chiral fields at the boundary leads to a simple group-theoretical formula for the intersections of ground states in the LG phase, which coincides with the geometrical intersections at large volume by a Hirzebruch-Riemann-Roch formula. This correspondence between group-theoretical data of the orbifold singularity at small volume and the geometric intersections of the compact cohomology at large volume is a well-known problem in mathematics, the McKay correspondence. We will identify the bosonic/fermionic topological bases $\{R_a\}/\{S^a\}$ with the dual bases for the compact/non-compact K-theory group on X, introduced by Ito and Nakajima in the case $\mathbb{C}^3/\Gamma$. Thus the open string point of view gives a completely natural explanation of the McKay correspondence in terms of the interpolation between the small and large volume phases of a two-dimensional quantum field theory, described by the group-theoretical structure of LG fields at small and exceptional sheaves at large volume, respectively.

The large volume representations of the bases $\{R_a\}$ and $\{S^a\}$ carry an interesting mathematical structure: they represent an exceptional collection of rigid sheaves that give a foundation of a so-called helix of exceptional sheaves [21, 22, 13]. This structure comes with a distinguished operation on the exceptional sheaves, the so-called mutation $\underline{R}$, and it turns out that the topological bases $\{R_a\}$ and $\{S^a\}$ are related by a specific series of such mutations, namely $S^{a\,*} = \underline{R}^{N-a}\, R_a$. This relation is quite remarkable, as it provides an effective way to define the bases $\{S^a\}$ in terms

[1] To simplify notation we will loosely refer to the small volume orbifold phase also as the Landau-Ginzburg phase, irrespectively of the choice of the superpotential.

[2] In fact many properties of the sector with boundary are similar to the bulk sector of the theory with $(0,2)$ world-sheet supersymmetry.

of short exact sequences starting from the basis $\{R_a\}$. One may then give yet another interesting representation of the topological bases in terms of the local mirror description of exceptional sheaves derived by Hori, Iqbal and Vafa [13]. In the LG theory mirror, the two bases are identified as the unique complete, supersymmetric D-brane configurations and related by a special monodromy.

Following the ideas of Diaconescu and Douglas [14], we may then use the D-branes on the Kähler manifold X to study D-branes on a generic, *compact* Calabi-Yau Y, by embedding Y in X as a hypersurface. We will find a simple relation between the data on X and Y that allows us to define the topological bases $\{R_a\}$ and $\{S^a\}$ on the ambient space or on the hypersurface interchangeably. This relation makes it possible to work even on a *singular* ambient space if the hypersurface avoids the singularities. This suggest a natural generalization of the McKay correspondence to singularities without a crepant resolution: as a relation between the boundary ring of small volume LG fields and their large volume sheaf counterparts on the smooth hypersurface of minimal codimension in the maximal crepant resolution of X provided by the LSM.

The topological bases $\{R_a\}$ and $\{S^a\}$ provide also an extremely effective description of the small/large volume D-branes on a generic, compact Calabi-Yau Y in terms of the simple, group theoretical data of the orbifold embedding space. This improves substantially on the cumbersome closed string methods used so far, namely analytic continuation of periods [8] and the toric construction of refs. [10, 14]. In particular we will show, how a surprising amount of information on the moduli space of Y, such as monodromy matrices and the analytic continuation of periods, may be derived with ease from the topological bases of the open string sector.

The organization of this paper is as follows: in section 2 we summarize the structure of the zero mode sector of the gauged linear sigma model with gauge group H with an emphasis on the sector with boundaries coupled to gauge fields. In section 3 we argue that the boundary conditions select naturally two bases of generators for the infinite dimensional zero mode sector with boundaries. The first, $\{R_a\}$, is obtained from the trivial ground-state by bosonic maps and is delocalized on the target space. The second, $\{S^a\}$ arises from fermionic maps and localizes on compact submanifolds. In section 4 we specify the two bases in the small volume, orbifold phase and define an inner product which is essentially the decomposition of tensor products of H representations in the multiplication of two-dimensional chiral matter fields. The inner product is non-degenerate on the two bases $\{R_a\}$ and $\{S^a\}$ and moreover the two are orthogonal to each other. In section 5, by a variation of the FI terms, we continue the objects R_a and S^a through the moduli space and obtain their large volume definitions as geometric sheaves. Specifically, the elements of $\{R_a\}$ and $\{S^a\}$ are identified as exceptional sheaves that generate the infinite space of sectors with boundaries in terms of complexes. We discuss also some properties of a general pairing of the closed and open string sectors for target spaces with non-negative first Chern class.

In section 6 we connect the previous ideas to the McKay correspondence. We identify the bosonic/fermionic bases with the generators of the non-compact/compact K-theory groups on quotient singularities, that have been introduced in the mathematical literature [23],[14]. In section 7 we argue that the bases $\{R_a\}$ and $\{S^a\}$ are foundations of a helix structure and related by a series of mutations. This leads to a simple description of the basis $\{S^a\}$ in terms of sequences involving the basis $\{R_a\}$. Moreover, using the results of [13], we identify the two bases as the unique complete, supersymmetric bases in the local mirror LG theory. In section 8 we turn to the case of Calabi-Yau hypersurfaces embedded in a target space with $c_1 > 0$. We find a simple relation between the K-theory data on the target space and the hypersurface and use it to formulate a natural proposal for a McKay correspondence in singular resolutions. Finally, in section 9 we apply the previous ideas to study D-branes on Calabi-Yau three-folds. We show that the intersection matrix of the basis $\{S^a\}$ agrees with that derived from the Gepner model and give an open string derivation of the topological data of the Calabi-Yau that enter the prepotential. On the base of an explicit example we expose the correspondence between localization of two-dimensional fermions and the fractional branes wrapped on submanifolds in Y. Moreover we show how the D-brane spectrum, monodromies and analytic continuation matrices may be derived from the group theoretical data of the orbifold.

In the appendices A, B, C we have collected a few simple examples to illustrate various aspects of the discussion.

2. Open strings and the gauged linear sigma model

2.1 The zero mode sector of closed and open strings

We consider the two-dimensional $(2,2)$ supersymmetric gauge theory with gauge group H and matter super-fields X_i in representations of H with an action of the form

$$S = S_{\mathrm{kin}} + S_{\mathrm{gauge}} + S_{FI} + S_W \,, \tag{2.1}$$

where the first two summands are the matter and gauge kinetic terms and the remaining terms are the superpotentials for the twisted and untwisted chiral fields, respectively. We refer to [16, 24, 13] for details and notation. In particular, [13] contains a profound study of theories of this type in the presence of boundaries, with boundary conditions on the fields that preserve 1/2 of the supersymmetry. For related work, see [6, 25, 26].

The primary interest in these theories is that the special class of conformal theories may describe the world-sheet theory of a type-II string for an appropriate field content. In this case, the boundaries correspond to BPS D-branes on which open strings may end. However, as in [13], many of the following considerations make sense more generally, also for the non-conformal case and target spaces of any

JHEP01(2001)018

dimension. We will loosely refer to the sector with and without boundaries also as the "closed" and "open" string sectors, respectively, though this nomenclature requires strictly speaking the above conditions to be satisfied.

An important example is the non-linear sigma model on an n-dimensional Kähler manifold X with $W = 0$ and no gauge fields. The lowest components x_i of the fields X_i represent coordinates on X and the kinetic terms in (2.1) are determined by the Kähler metric $g_{i\bar{j}}$ on X. The left- and right-moving fermions $(\psi^i_+, \psi^{\bar{i}}_+)$ and $(\psi^i_-, \psi^{\bar{i}}_-)$ in the super-fields X_i are sections of $K^{1/2} \otimes (\Omega^{*\,(1,0)} \oplus \Omega^{*\,(0,1)})$ and $K^{1/2\,*} \otimes (\Omega^{*\,(1,0)} \oplus \Omega^{*\,(0,1)})$, respectively. Here K is the canonical bundle on the 2d world-volume Σ and $\Omega^{1,0}$ $(\Omega^{0,1})$ denotes the pull-back of the (anti-)holomorphic cotangent bundle of X to Σ.

The zero mode sector of this sigma model without boundaries has a topological nature. The ground states in the RR sector are in one-to-one correspondence with the Dolbeaut cohomology on X [18, 19]. The four supercharges act as the Dolbeaut operators ∂, $\bar{\partial}$ and their adjoints on the Hilbert space $\mathcal{H}^{\mathrm{top}}_{\mathrm{cl}}$ which can be identified with the space of sections of

$$(\wedge\Omega^{(1,0)}) \wedge (\wedge\Omega^{(0,1)})\,. \tag{2.2}$$

By spectral flow the RR ground states are related to the so-called (ac) ring of primary chiral fields in the NS^2 sector [20]. It is a deformation of the cohomology ring defined by wedging forms in $H^*(X)$. For this reason this ring is also called the quantum cohomology ring. If $c_1(X)$ is zero, there is another correspondence between the supersymmetric ground states and the elements of the so-called (cc)-ring. We will mostly focus on the (ac)-ring, however.

The Hilbert space in the open string sector corresponds to the addition of boundaries and has again two sectors, denoted as the A-type and B-type boundary conditions, respectively [6, 13]. They are naturally associated with the (cc)- and (ac)-ring of the closed string sector. The boundary condition sets to zero the two linear combinations of the four fermionic zero modes that correspond to dz^i and $i_{\partial/\partial z^i}$ [13]. Moreover the fermions on the open string are coupled to the gauge field on the boundary D-brane. The topological sector of the open string Hilbert space $\mathcal{H}^{\mathrm{top}}_{\mathrm{op}}$ is therefore identified with the space of sections of

$$\wedge\,\Omega^{(0,1)} \otimes E^*_a \otimes E_b\,, \tag{2.3}$$

where E_a and E_b are the gauge bundles that correspond to the gauge fields on the two D-branes labeled by a and b, on which the open string ends. The two unbroken supercharges include the gauge fields and act on the Hilbert space as the Dolbeaut operator $\bar{\partial}_A = d\bar{z}^{\bar{i}}(\partial_{\bar{i}} + A^{(b)}_{\bar{i}} - A^{(a)}_{\bar{i}})$ and its adjoint, respectively.

Note that, contrary to (2.2), the total space (2.3), is infinite dimensional.

2.2 Open strings in the gauged linear sigma model

To discuss open strings we need to consider gauge fields. We will argue that a minimal extension of the sigma model by gauge fields, namely Witten's gauged linear sigma model construction of a Kähler manifold X as a coset G/H' [16], contains the necessary degrees of freedom to define a finite basis B that generates $\mathcal{H}_{\rm op}^{\rm top}$, in a sense to be made precise.

The gauge group H of the gauged LSM is a subgroup of H'. The matter fields X_i carry representations of H (and also of the global symmetry G) and their scalar components x_i represent homogeneous coordinates on X. Their fermionic super-partners ψ^i take values in the tangent bundle of X, after taking into account the identifications made by the H gauge invariance. Examples considered in [16] include flag manifolds with $H' = \prod_i U(n_i)$ and toric varieties with $H' = U(1)^r$. We will discuss in detail mainly the case where the gauge group $H \subset H'$ is abelian, though generalizations are possible and will be commented on along the way.

The ground states of the gauged LSM have a representation of the form

$$f(x_i)\, \psi^{\alpha_1} \wedge \cdots \wedge \psi^{\alpha_n}\,, \tag{2.4}$$

where ψ^α denotes any fermionic zero mode and $\alpha \in \{i, \bar{i}\}$. As the fields X_i carry H representations, the state corresponding to (2.4) will be a section of some bundle V which is determined by the tensor product of the H representations and the (anti-)holomorphic tangent indices of the fermions. We identify V as the "difference bundle" $E_a^* \times E_b$ of an open string sector between boundaries carrying the gauge bundles E_a and E_b, respectively. In the closed string sector without boundaries, V is trivial and the function $f(x_i)$ is determined, up to total derivatives, by the ordinary Dolbeaut operator. On the other hand, in the sector with boundaries, there will be an generically infinite number of allowed functions $f(x_i)$ corresponding to an infinite number of different V valued Dolbeaut operators.

Clearly we can generate only a subset $\mathcal{H}_{\rm op}^{\rm top,0} \subset \mathcal{H}_{\rm op}^{\rm top}$ in this way, as the available bundles V are constrained by the representations of the fields X_i, and so are, loosely speaking, a combination of H bundles and the tangent bundle. More precisely, the fermions contain the information not only about bundles on X but also sheaves supported on holomorphic submanifolds. The important point is that the bundles (or sheaves) that have a representation (2.4) in the linear sigma model will turn out to be sufficiently general to generate a finite basis B for the infinite dimensional Hilbert space $\mathcal{H}_{\rm op}^{\rm top}$. In geometric terms, the available sheaves will be generators for the derived category $D^\flat(X)$, which means that any coherent sheaf on X may be constructed in terms of (bounded) complexes of elements in B.

3. Two dual bases for $\mathcal{H}_{\rm op}^{\rm top}$ and localization properties

As the space $\mathcal{H}_{\rm op}^{\rm top}$ (2.3) is infinite dimensional it is not obvious in the geometrical large volume phase what a finite basis for it might look like. The key point will be the construction in the small volume phase, which leads to a definite recipe for a construction of two dual, finite bases of generators with dimension $N = \dim H^{\text{vert}}(X)$, where $H^{\text{vert}}(X) = \oplus_k H^{k,k}(X)$.

Similar as we may build up the closed string sector $\mathcal{H}_{\text{cl}}^{\text{top}}$ from the ground state $\mathbf{1}$ by acting with the fermionic zero modes on it, we start in the open string sector from the ground state $\mathcal{O}$, corresponding to a section of the trivial bundle in the large volume limit. In fact the choice of the "base point" $\mathcal{O}$ is irrelevant and it may be replaced by any line bundle $\mathcal{O}(n_0)$, as the difference corresponds to a monodromy, or, equivalently, to a change of the closed string background.

From (2.4), we expect that in some sense, we may obtain the sections in $\mathcal{H}_{\text{op}}^{\text{top}}$ by multiplying sections of $\mathcal{O}$ with bosonic and fermionic zero modes. The world-sheet point of view provides a more concrete approach. The B-type boundary conditions of the $(2,2)$ supersymmetric sigma model (without B-field) are are [6, 13, 27, 28]:

$$\partial_1 \phi^\theta = \psi_-^\theta - \psi_+^\theta = 0 \,, \qquad \partial_0 \phi^n = \psi_-^n + \psi_+^n = 0 \,, \tag{3.1}$$

where θ and n are indices in the tangent and normal directions, respectively. The conserved supercharge is

$$Q = \sqrt{2} \int g_{i\bar{j}} \left((\psi_+^{\bar{j}} + \psi_-^{\bar{j}})\, \partial_0 \phi^i + (\psi_-^{\bar{j}} - \psi_+^{\bar{j}})\, \partial_1 \phi^i \right) . \tag{3.2}$$

The representation theory of the supercharge Q on the boundary has not been worked out but we will need only the following, simple observation. On the boundary, Q splits into the two parts

$$\begin{aligned} Q_{\text{tangent}} &= \sqrt{2} \int g_{\theta\bar{\theta}}\, (\psi_+^{\bar{\theta}} + \psi_-^{\bar{\theta}})\, \partial_0 \phi^\theta \,, \\ Q_{\text{normal}} &= \sqrt{2} \int g_{n\bar{n}}\, (\psi_-^{\bar{n}} - \psi_+^{\bar{n}})\, \partial_1 \phi^n \,. \end{aligned} \tag{3.3}$$

It follows that the super-fields on the boundary have a structure similar to that of $(2,0)$ super-fields, with bosonic multiplets in the tangential and fermionic multiplets in the normal directions. Specifically, the components of the $(2,2)$ fields $\Phi^i = \phi^i + \sqrt{2}\theta^+\psi_+^i + \sqrt{2}\theta^-\psi_-^i + 2\theta^+\theta^- F^i + \cdots$ that survive the boundary conditions may be assembled into super-fields of the form $\phi^\theta + \sqrt{2}\theta'\psi^\theta$ and $\psi^n + \sqrt{2}\theta' F^n$, respectively, with θ' the parameter for the surviving supersymmetry (3.2).

We choose now the base point $\mathcal{O}$ in the infinite dimensional space (2.3). As $\mathcal{O}$ is spread over all of X, the boundary conditions are of Neumann type in all directions in this sector. Multiplication of this state with the fields Φ^θ yields another ground state with identical spatial boundary conditions, but different gauge bundle V, determined by the H representation of Φ^θ. On the other hand we may also consider a sector with a new Dirichlet boundary condition on a submanifold C in X. In this sector, the lowest component of the LG field Φ^n is *fermionic* in the normal directions. Note

JHEP01(2001)018

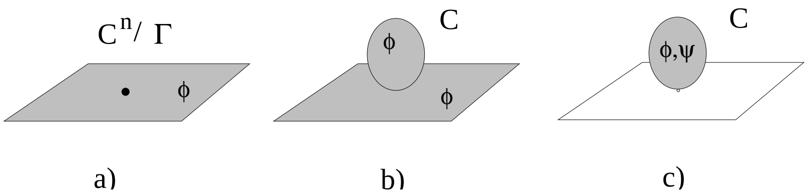

Figure 1: *a*) The ground state $\mathcal{O}$ for the trivial bundle on the non-compact space $\mathbb{C}^n/\Gamma$. All directions are tangential and the projection of a chiral multiplet at the boundary yields a bosonic super-field; *b*) the resolution with compact exceptional divisor C with the same boundary conditions; *c*) the boundary condition that corresponds to a D-brane on C projects onto fermionic super-fields in the normal and bosonic super-fields in the tangential directions.

that the fermionic degrees of freedom are confined to the boundary in the normal directions. Clearly this corresponds to a D-brane localized on C.

We illustrate this in figure 1 for the type of geometry that we will focus on in later sections, namely the resolution of a quotient singularity $\mathbb{C}^n/\Gamma$. The ground state $\mathcal{O}$, corresponding to the Neumann boundary condition in all directions, projects onto bosonic super-fields. In the space-time sense, the D-brane that corresponds to this sector has infinite mass due to the non-compactness of the space. In the resolution with a compact exceptional divisor C, there are new boundary conditions that correspond to a finite mass D-brane on a compact cycle C. The projection of a $(2,2)$ multiplet at this boundary adds a fermionic super-field that lives on C and projects out the boson in the normal directions.

To construct bases of $\mathcal{H}_{\rm op}^{\rm top}$, we may simply reverse the logic and note that multiplication of $\mathcal{O}$ by the lowest component of a $(2,2)$ super-field leads to a sector with a different bundle but the same spatial boundary conditions. On the other hand multiplication by a fermionic zero mode ψ^i corresponds to changing the boundary condition from Neumann to Dirichlet in the directions normal to the hyperplane on which ψ^i is localized. In the next section we construct two bases $\{R_a\}$ and $\{S^a\}$ in the LG phase obtained by multiplication of $\mathcal{O}$ with only bosonic or only fermionic zero modes respectively. We will subsequently study some remarkable properties of these bases and eventually show that they provide good finite bases of generators for $\mathcal{H}_{\rm op}^{\rm top}$. In agreement with the above localization arguments, the large volume version of the two bases $\{R_a\}$ and $\{S^a\}$ will correspond to bases for the general K-theory group $K(X)$, and the K-theory group $K_c(X)$ with compact support, respectively. This leads to a beautiful identification of the bosonic/fermionic bases as dual bases of a McKay correspondence, as introduced in the mathematical literature by Ito and Nakajima [23].

4. Linear sigma model I: the group theoretical perspective

Witten's gauged linear sigma model [16, 17] description of the Kähler manifold $X = G/H'$ is a (2,2) theory of the form (2.1) with canonical kinetic terms, a gauge group

$H \subset H'$ and matter fields X_i in representations of H. As will be reviewed in the next section, there are two distinguished types of phases of this theories controlled by the FI terms, corresponding to small and large volume, respectively. We will first consider the small volume phase which corresponds geometrically to some orbifold $\mathbb{C}^{n+1}/\Gamma$, with Γ a discrete subgroup of H. It carries a natural group theoretical structure. In the presence of a superpotential W, which we will add later, this phase describes a Landau-Ginzburg theory. We will first construct two bases $\{R_a\}$ and $\{S^a\}$ which are our candidates for a finite basis of generators for $\mathcal{H}_{\rm op}^{\rm top}(X)$ in this phase. In the next section we carry the bases to large volume, by varying the FI terms, and argue that the necessary conditions for them to represent a basis of free generators are satisfied.

4.1 The case $H = \mathrm{U}(1)$

Let us start with the linear sigma model with gauge group U(1) and $n+1$ matter fields X_i of charges $w_i \in \mathbb{Z}$. As in [16] it will be natural to extend the gauge group U(1) to $\mathbb{C}^*$. Its action on the scalar fields is given by $x_i \to \omega^{w_i}\, x_i$, with $\omega \in \mathbb{C}^*$. If the weight vector $w = (w_1, \ldots, w_{n+1})$ has only positive entries, X is a compact weighted projective space $\mathbb{WP}^n_{w_i}$.

If the weights w_i are all equal, this space is smooth and the above theory will also describe the geometric large volume phase for appropriate values of the FI terms. Otherwise, X will have singularities at the fixed points of the $\mathbb{C}^*$ action that have to be resolved to obtain a smooth space. The resolution requires the addition of extra matter fields and U(1) factors and will be discussed in the next section.

4.1.1 A basis $\{R_a\}$ from bosonic maps

Let us first consider multiplication of the state $\mathcal{O}$ with the bosonic components of the LG fields, that is the homogeneous coordinate ring. As the fields x_i carry only U(1) charges, these states will flow to a basis of line bundles in the large volume phase. We denote a state with U(1)-charge q obtained in this way q by $\mathcal{O}(q)$:

$$x_i : \mathcal{O} \to \mathcal{O}(w_i)\,. \tag{4.1}$$

As discussed already, we may shift the origin from $\mathcal{O}$ to $\mathcal{O}(-\infty)$. In this way we obtain an infinite series of states with U(1) charge $q \in \mathbb{Z}$. Let $\mathcal{H}_R = \{\mathcal{O}(q)\},\ q \in \mathbb{Z}$ denote this infinite set, ordered with increasing U(1) charge.

4.1.2 A dual basis $\{S^a\}$ from fermionic maps

Instead of multiplying a ground state $\mathcal{O}(n_0)$ with x_i we may consider, in view of (2.4), multiplication with the fermionic zero modes ψ^i:

$$\mathcal{O}(n_0) = S^1 \xrightarrow{\psi[q]} S^{q+1}\,, \tag{4.2}$$

where $\psi[q]$ is a product of fermions with U(1) charge q. We fix our conventions such that the creation operator corresponds to a fermion that is a section of the tangent bundle; thus the objects S^a live in the space dual to (2.3). Different then before, the composition of fermionic maps is anti-symmetric and the construction terminates at charge $q = N$, which is the charge of the product of all fermions. In fact this combination of zero modes is equivalent to 0 by U(1) gauge equivalence. Thus we get naturally a vector of N elements $\mathcal{E}_S = \{S^a\} = \{S^1, \ldots, S^N\}$. Note that there may be values $q = a'$, where no map (4.2) exists and we may not construct the sector $S^{a'}$ with charge a' relative to S^1. These ground states missing in the orbifold phase will be recovered in the resolution of the orbifold, which introduces extra matter fields Z_i. In particular, in the resolution, the maps $S^1 \to S^a$ exists for all values of a, with those missing in the orbifold phase provided by the fermions in the new super-fields Z_i.

We can again define an infinite set $\mathcal{H}_S$ of bundles, consisting of an infinite number of copies of $\mathcal{E}_S$ with origin shifted by N units of U(1) charge. This is ordered set is dual to $\mathcal{H}_R$ w.r.t. the bilinear product defined in the next section. The fermionic zero modes carry also an index of the tangent bundle. Therefore, the bundles in the large volume phase connected to the states in $\{S^a\}$ may have rank larger than one. The construction of these bundles in terms of sequences will be discussed in section 5 after we have described the smooth resolution in the large volume phase.

4.2 The Witten index

The weighted number of closed string ground states (2.2), the Witten index $(-1)^F$ [18], equals $\chi(X)$. The index in the open string sector ab coincides, in virtue of (2.3), with that of $\bar\partial_A$, which, for a smooth space, is described by the Hirzebruch-Riemann-Roch formula:

$$\mathrm{ind}_{\bar\partial_A} = \sum_k (-1)^k \ \mathrm{Ext}^k(E_a, E_b) \overset{HRR}{=} \int_X \mathrm{ch}(E_a^*) \ \mathrm{ch}(E_b) \ \mathrm{td}(X)\,. \tag{4.3}$$

Motivated by the first expression, let us define an inner product $\langle A, B\rangle_H$ on elements in $\mathcal{H}_{\mathrm{op}}^{\mathrm{top},0}$ as the number of holomorphic maps f from A to B. For the ordered set $\mathcal{H}_R$, the maps $f_{a,b}$ are bosonic of degree $b - a$ and their number is equal to the number of independent monomials in the homogeneous coordinate ring with this degree. For a reason that will become clear momentarily we restrict to a basic set $\mathcal{E}_R$ of $N = \sum_i w_i$ consecutive elements in $\mathcal{H}_R$. Note that $N \cdot K$ is the first Chern class of X. The index for the elements $R_a \in \mathcal{E}_R$ is

$$\chi_{ab}^H \equiv \langle R_a, R_b\rangle_H = \Big(\prod_i \sum_{k=0}^{N-1} h^{k\,w_i}\Big)_{ab} = \Big(\prod_i (1 - h^{w_i})^{-1}\Big)_{ab}\,. \tag{4.4}$$

Here h, is the $N \times N$ shift matrix with unit entries above the diagonal and zeros otherwise; it fulfills $h^N = 0$. The formula (4.4) contains only the group theoretical information of H and is, contrary to (4.3), well-defined even if X is singular. It will coincide with r.h.s. of (4.3) on a smooth resolution $\tilde{X} \to X$ with the R_a defined as the appropriate pull backs[3] to $\tilde{X}$. We identify the degree k with the fermion number of the map and thus the only contribution to the index comes from $k = 0$.

Similar we may determine the inner product on the set $\mathcal{E}_S$, where the maps carry non-trivial fermion numbers $k = 0, \ldots, N-1$. The counting formula for these maps is the same as in the bosonic case, up to an extra minus sign for each monomial x_i and we obtain

$$\chi^{H\,ab} = \langle S^a, S^b \rangle_H = \Big(\prod_i (1 - h^{w_i}) \Big)^{ab} . \tag{4.5}$$

Note that the basis $\mathcal{E}_R = \{R_a\}$ from bosonic maps and the dual $\mathcal{E}_S^* = \{S^{a\,*}\}$ of the basis from fermionic maps are orthogonal with respect to the inner product $\langle A, B \rangle_H$, if the "base point" matches. In fact it follows from (4.4) and (4.5) that the duals of the elements in the set $\mathcal{E}_S = \{S^1, \ldots, S^N = \mathcal{O}(-n_0)\}$ fulfill $\langle S^{a\,*}, R_b \rangle_H = \delta^a_b$. The $S^{a\,*}$ may be formally written as the linear combinations

$$S^{a\,*} = \chi^{H\,ab} R_b \,. \tag{4.6}$$

In particular, eq. (4.6) describes the relation between the Chern characters of the two dual bases $\{R_a\}$ and $\{S^a\}$ on a smooth resolution $\tilde{X}$.

4.3 Generalizations to other gauge groups H

The generalization of the above ideas for general $X = G/H'$ appears to be relatively straightforward. The matter fields X_i come in representations r_H of $H \subset H'$ and describe more general world volume gauge theories of D-branes wrapped on X. We may again define bosonic and fermionic maps, leading to representations generated by tensor products of r_H and the conjugate representations $\bar{r}_H$, respectively. The requirement that the elements in a basis B represent free generators of $D^\flat(X)$ imposes a non-trivial selection rule on the allowed representations in B.[4]The two dual bases $\{R_a\}$ and $\{S^a\}$ constructed in this way will again satisfy an orthogonality relation $\langle S^{a\,*}, R_b \rangle_H = \delta^a_b$.

5. Linear sigma model II: the geometric perspective

5.1 From small to large volume

Consider the $H = \mathrm{U}(1)$ theory with $n+1$ fields X_i of positive charges w_i, now with one extra field P of negative charge $-N$, where $N = \sum_i w_i$. As the sum of all charges

[3]This will be further discussed below.

[4]We will formulate a conjecture for a group theoretical version of this selection rule in section 6.

is equal to zero, this a CFT. A supersymmetric vacuum of the theory must satisfy the D-term equations

$$\sum_i w_i \, |x_i|^2 - N \, |p|^2 - r = 0 \,, \tag{5.1}$$

where r is the FI parameter of the U(1). A variation of the FI term interpolates between the geometric and the LG Higgs phases of the 2d QFT [16].

For positive r, at least one of the x_i has to be non-zero. The space parameterized by the x_i divided by the U(1) action (together with a careful treatment of the singular orbits [16, 17]) gives the symplectic quotient construction of the weighted projective space $X = \mathbb{WP}^n_{w_i}$. The scalar p is a coordinate on the bundle $\mathcal{O}_X(-c_1(X))$. The total space of this bundle is an $n+1$ dimensional Calabi-Yau, non-compact in the p-direction. This is the geometric Higgs phase where the U(1) gauge symmetry is spontaneously broken by the vev's of the x_i. The open string, or D-brane, states in this phase may be interpreted as elements of the K-theory on X [29]–[31].

As r is decreased to negative values, the size of X shrinks to zero (at least classically). From (5.1), we see that the scalar field p must be nonzero. As p has charge N, the H gauge symmetry is broken by the vev of this field to the residual, discrete gauge symmetry $\Gamma = \mathbb{Z}_N$. There are two different branches according to the values of the fields x_i. At $x_i = 0$, there is an unbroken gauge symmetry $\Gamma \in H$. For $x_i \neq 0$, also the subgroup Γ is broken. As p varies, the values of x_i parametrize the geometric orbifold $\mathbb{C}^{n+1}/\Gamma$. This space is again a non-compact Calabi-Yau, with a quotient singularity at the origin.

Similar as the variation of the FI parameter r leads to a interpolation between LG and the geometric large volume phase in the closed string sector, it connects the ground states of the topological sector with boundaries in the two phases. Specifically, the states R_a and S^a constructed previously are connected to sheaves in the large volume phase. In the following we continue these states to their large volume counterparts on the smooth resolution and argue that they generate $\mathcal{H}^{\rm top}_{\rm op}$. In particular in the large volume the multiplication with x_i and ψ^i is interpreted as a the multiplication ring of sections of (2.3).

5.2 Ring structures, a pairing and the finite basis of generators

Before we proceed with an explicit construction of the finite bases $\{R_a\}$ and $\{S^a\}$ of generators as bundles at large volume, let us discuss in which sense they will generate the infinite dimensional space $\mathcal{H}^{\rm top}_{\rm op}$. Moreover we consider some interesting properties of a natural topological pairing with the closed string sector, which determines, once again, the dimension of $\{R_a\}$ and $\{S^a\}$.

Let us assume for now that we work on a smooth space X, and the bases $\{R_a\}$ and $\{S^a\}$ are defined as sheaves on X. The inner product (4.4) is non-degenerate, as is obvious from the second expression. It follows that the basis $\{R_a\}$ generates the elements of the diagonal closed string Hilbert space $H^{\rm vert}(X)$ by its Chern classes.

The same is obviously true for the basis $\{S^a\}$. This fixes in particular the dimension $N = \dim H^{\text{vert}}(X)$ of $\{R_a\}$ and $\{S^a\}$.

Similarly, the Chern classes of the bases $\{R_a\}$ or $\{S^a\}$ generate the Chern classes of all open string states by linear combinations. A stronger requirement on a true basis B of generators is that bounded complexes of elements of B generate all sheaves on X. A necessary condition on B is that there are no higher Ext groups between the elements in B and that they provide free generators of the homotopy category $D^\flat(X)$ of finite complexes of sheaves on X. The first property is obvious for the basis $\{R_a\}$, as the only contribution to the index (4.3), comes from $k = 0$. It is in this sense that the basis $\{R_a\} \subset \mathcal{H}_{\text{op}}^{\text{top},0}(X)$ generates $\mathcal{H}_{\text{op}}^{\text{top}}(X)$.

The above is also true, though less obvious, for the basis $\{S^a\}$. The reason is that the grading of the extension groups changes along the flow from small to large volume [12].[5]This will be further discussed on the basis of the explicit construction of the sheaves S^a.

Let us consider now an interesting topological pairing between the open and closed string states provided by the Chern character. For a B-type boundary state $A \in H^p(X, V = E_a^* \otimes E_b)$ and a closed string state in the vertical cohomology $\eta \in H^{\text{vert}}(X)$, we may consider the integral

$$(A, \eta) = \int_{\tilde{\eta}} \text{ch } A\,, \tag{5.2}$$

where $\tilde{\eta} \in H_{k,k}(X)$ is the dual of η and we use A also to represent a section of the corresponding bundle. We could have defined other parings that include non-trivial topological invariants of X under the integral. Recall that the vertical cohomology $H^{\text{vert}}(X)$ comes with a distinguished, integral ring structure, namely the quantum cohomology ring [20]. Similarly, we expect a quantum ring structure to be defined on $\mathcal{H}_{\text{op}}^{\text{top}}$ along the lines of [2]. We should therefore look for the distinguished pairing between the open and closed string Hilbert spaces that respects *integral* ring structures. Inspired by the integrality of the inner product (4.3), we consider its "square root"

$$Q^{(A)} = \text{ch}(A)\sqrt{\text{td}\, X}\,. \tag{5.3}$$

This defines a charge $Q^{(A)} \in \mathcal{H}_{cl}$ which is the K-theoretic version [32] of a formula obtained for the macroscopic RR-charge of a stringy D-brane by anomaly considerations [33, 34]. However (5.3) makes sense more generally in the non-conformal (2,2) sigma model and with a target space of any dimension. The charge $Q^{(A)}$ defines a specific pairing $(A, \eta) = \int_{\tilde{\eta}} Q^{(A)}$. The index (4.3), rewritten in terms of $Q^{(A)}$ becomes

$$Q^{(A)} \cdot Q^{(B)} \equiv \langle A, B\rangle = \int_X \Big(\sum_k (-1)^k\, Q^{(A)}|_{k,k}\Big)\, Q^{(B)}\, e^{c_1(X)/2}\,, \tag{5.4}$$

[5]It is also easy to see, that the flow does not change the grading of the extension groups between the elements in $\{R_a\}$, so that they remain a good basis also at large volume.

where the subscript $|_{k,k}$ denotes taking the (k,k)-form part of an expression. Note that all higher Chern classes of X cancel out of this expression so that it depends only on $c_1(X)$.

The bilinear form (5.4) is defined on the infinite dimensional space $\mathcal{H}_{\rm op}^{\rm top}(X)$. If $c_1(X)=0$, the expression (5.4) is symmetric (anti-symmetric) and displays the orthogonal (symplectic) structure of the intersection form on the even (odd) dimensional Calabi-Yau X; it coincides with it when restricted to a finite basis. In particular, for n odd, χ_{ab} becomes the Dirac-Zwanziger product in the space-time gauge theory obtained by a type-IIA compactification on X. For $c_1(X)\neq 0$, the form $Q^{(A)}\cdot Q^{(B)}$ still defines an integral bilinear product on the Hilbert space $\mathcal{H}_{\rm op}^{\rm top}(X)$, but has no symmetry properties.[6]

We have not considered a relation between the (cc) and A-type boundary states, which in the Calabi-Yau case provides a mirror description of the above states on the mirror manifold. A natural pairing between open and closed string states has been studied in [36, 37, 6, 24, 13]. In a particular sector, it is related to the period integrals on $\tilde{X}$. It would be interesting to study the precise relation between this pairing and the one defined by (5.3). We will comment on this connection in more detail in section 7.

5.3 A basis of line bundles $\{R_a\}$ in the geometric phase

If the weights w_i are equal, X is smooth and we can interpolate the states $\mathcal{O}(q)$ from small to large volume without further modifications. As the lowest bosonic components x_i are sections of line bundles, the states $\mathcal{O}(q)$ will correspond to line bundles of Chern class $q\cdot K$ on X, with K the hyperplane class of X. In particular eq. (4.1) is naturally interpreted as the multiplication of sections in $\mathcal{H}_{\rm op}^{\rm top}(X)$.[7]

For general w_i, the $\mathbb{C}^*$ action $x_i\to\omega^{w_i}x_i$ has fixed points that lead to singularities on X. To define the states $\mathcal{O}(q)$ properly as line bundles we need to specify their sections on a smooth resolution $\tilde{X}$ of X. We proceed to construct this bundles on

[6] In contrary, the Dirac index $\mathrm{ind}_{\not{\partial}_A}=\int_X \mathrm{ch}(E_a^*)\,\mathrm{ch}(E_b)\,\hat{A}(X)=\int_X(\sum_k(-1)^k\,Q^{(a)}|_{k,k})\,Q^{(b)}$, with $\hat{A}(X)=e^{-c_1(X)/2}\,\mathrm{td}(X)$ the A-roof genus, has good symmetry properties, but needs not to be integral in the given basis B. There is then an obstruction to define the corresponding bundle on X. The fact that the index (4.3) and the Dirac index differ is related to the anomaly of the $(2,2)$ sigma model for $c_1(X)=0$, as studied in detail in [35], see also [34] for a related work.

[7] Recall the relation between line bundles and divisors. A line bundle $\mathcal{L}$ is characterized by its first Chern class $[L]\in H^{1,1}(X,\mathbb{Z})$, which is dual to an algebraic submanifold of codimension one, a divisor $L\in H_{n-1,n-1}(X)$. The divisor L is locally defined as the zero of a meromorphic function f in the homogeneous coordinates x_i. The first Chern class of the line bundle associated to L is determined by the weights of f w.r.t. to $r=h_{1,1}$ weight vectors w^α that represent the classes $H^{1,1}(X)$. In fact r is precisely the number of U(1) factors in H and the vectors w^α describe the charges of the matter fields X_i under the gauge group $\mathrm{U}(1)^r$. For the toric varieties we consider, the line bundles are indeed in one-to-one correspondence with monomials in the coordinates x_i, as the toric divisors $D_i: x_i=0$ are known to span the Picard lattice of X [38].

a given (not necessarily unique) resolution of X. Note that, independently of the chosen resolution, the geometric inner product (4.3) on the basis of bundles $\{R_a\}$ will coincide with the group theoretical formula (4.4).

In the following we assume $n > 1$ to avoid complications special to the low dimensional cases. We will also consider only a single resolution of X; additional resolutions may be treated step for step. The linear sigma model in the new phase that corresponds to a partial resolution $\hat{X}$ has an $\mathrm{U}(1)^2_{\hat{X}}$ gauge symmetry and one extra matter field Z. The size of the divisor introduced in this resolution is a new FI parameter r'. The original $\mathrm{U}(1)_X$ symmetry of the phase corresponding to X is the linear combination of $\mathrm{U}(1)^2_{\hat{X}}$ under which the field Z is uncharged.

It is clear that any line bundle on $\hat{X}$ corresponds to a well-defined representation of $\mathrm{U}(1)_X$; however the map from $q_{\hat{X}}$ to q_X has the charge of the field Z as its kernel. To reconstruct the basis $\{R_a\}$ on $\hat{X}$ from that on X we require that *each map $R_a \to R_b$ generated by the field $x_i \in X_i$ in the phase X pulls back to a map in the new phase $\hat{X}$*. This determines uniquely the Chern class of the bundle $\hat{R}_a$ on $\hat{X}$.

E.g. , if $R_N = \mathcal{O}$ and x_0 a charge one field in the LG phase that provides a map $R_{N-1} \to R_N$, where $R_{N-1} = \mathcal{O}(-1)$ in the orbifold phase, then the Chern class of R^{N-1} in the large volume phase is $c_1(R_{N-1}) = -\sum_{i=1}^r q_0^\alpha K_\alpha$. Here q_0^α are the charges of the field $X_0 = (x_0, \dots)$ in this phase and the K_α the $(1,1)$ forms that correspond to the $\mathrm{U}(1)^r$ symmetry of the LSM for the resolution $\tilde{X}$. In other words, $\tilde{R}_{N-1} = \mathcal{O}(-q_0^1, \dots, -q_0^r)$, and similarly for the other bundles $\tilde{R}_a$. For an explicit example, we refer to section 9.3.

5.4 The dual basis $\{S^a\}$

From the previous considerations it is rather evident what kind of objects the S^a are in the geometric phase: the sheaves of sections generated by multiplication of a section of the line bundle $\mathcal{O}(n_0)$ with fermionic zero modes. The massless fermions of the LSM are described by the exact sequence:

$$0 \longrightarrow \mathcal{O}^r \longrightarrow \oplus_{i=1}^{n+r} \mathcal{O}(q_i^1, \dots, q_i^r) \longrightarrow \Omega^* \longrightarrow 0\,, \tag{5.5}$$

where $H = \mathrm{U}(1)^r$ is the gauge group of the LSM on the resolution $\tilde{X}$ and q_i^α the charges of the $n+r$ matter fields X_i and Z_i. The above is just the statement that their fermionic components are sections of the tangent bundle of $\tilde{X}$ and carry $\mathrm{U}(1)^r$ charge q_i^α.

In particular, if the weights are equal, $w_i = 1$, then the sheaves S^a are, by construction, simply the a-th exterior power of the twisted tangent bundle

$$S^a = (-)^{N-a}\, \Lambda^{a-1}\Omega^* \otimes \mathcal{O}(-n_0 - a)\,. \tag{5.6}$$

Here we have included a minus sign from the fermion number of the map in the definition. Note that $S^N = \Lambda^{N-1}\Omega^*(-n_0 - N)$ is the line bundle $\mathcal{O}(-n_0)$; the dual

JHEP01(2001)018

basis $\{S^{a\,*}\}$ is thus orthogonal to the basis of line bundles with $R_N = \mathcal{O}(n_0)$ in the orbifold phase. By construction, the exact sequence associated to the bundles S^a is simply the appropriate exterior power of (5.5)!

In general, if the w_i are not equal, the structure is similar and may be inferred from the two-dimensional point of view described in section 3. Roughly speaking, *the objects S^a represent exterior powers of the tangent bundles on compact, holomorphic submanifolds in $\tilde{X}$*, including $\tilde{X}$ itself.

As discussed above, the resolution $\tilde{X}$ of the singularities of X introduces new matter super-fields Z_i, $i = 1, \ldots, r-1$ together with $r-1$ new U(1) gauge multiplets. The former are associated with the exceptional divisors of the resolution defined by $D_i : z_i = 0$. Moreover, their fermionic super-partners ζ^i generate new Dirichlet boundary conditions along D_i and intersections thereof.

It is relatively straightforward to proceed from this general arguments to a more detailed description of the sheaves S^a in a concrete example. In section 9.3 we will study in detail a representative case that demonstrates the nice correspondence between the localization of the two-dimensional fermions and that of the fractional branes S^a, in complete agreement with the above picture.[8] Moreover, we will find a convenient closed form for the sequences that describe the S^a in section 7 in terms of mutations of exceptional collections. Here we restrict to outline the general structure from the two-dimensional world-sheet point of view.

We choose S^1 to be the trivial bundle $\mathcal{O}$ and assume that there are $k+1$ fermions ψ^i of charge 1. Then the sheaf S^2 is defined by the sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathbb{C}^{k+1} \otimes \mathcal{O}(1) \longrightarrow S^2 \longrightarrow 0\,, \tag{5.7}$$

and it has rank k. Here $\mathcal{O}(a)$ denotes the $a-$th power of the hyperplane bundle and we have assumed that we may eliminate the extra matter fields Z_i introduced in the resolution in this step. The bundle S^1 describes the pull back of Ω_E^* to X, where E is a smooth, holomorphic submanifold of dimension k in X, parametrized by the bosonic super-partners of the ψ^i. If $k > 2$, then S^3 is given by the sequence

$$0 \longrightarrow S^2 \longrightarrow \mathbb{C}^{(k+1)k/2} \otimes \mathcal{O}(1) \longrightarrow S^3 \longrightarrow 0\,. \tag{5.8}$$

Thus S^3 is the pull back of the second exterior power $\wedge^2\Omega_E^*$, and so on. At the $k+1$-th step this procedure terminates as $\wedge^{k+1}\Omega_E^*$ does not exist. At this point there will be a new set of fermions of charge $k+1$. If this set contains again some of the fermions ψ^i that are already present in the orbifold phase, the series of bundles S^a continues with another set of pull backs of (possibly twisted) tangent bundles on a submanifold in X.

A new situation arises if at some point there is no such fermion ψ^i, or it is equivalent to zero by the gauge invariance. In particular this happens if there is no

[8] Another instructive example is given in appendix B.

map $\psi[a] : S^1 \to S^a$ in the orbifold phase. In the resolution there will then be an additional fermion ζ^i of the appropriate charge. It is the fermionic component of one of the extra fields Z_i and the Dirichlet boundary condition imposed by it sets the bosonic component z_i to zero. The submanifold in X defined by this zero is an exceptional divisor $D_i : z_i = 0$. As the sheaf S^a does not involve other fermions that live on $\tilde{X}$, there are no non-zero sections away from the divisor $z_i = 0$. In other words, S^a is a sheaf on D_i, extended by zero on X. As the codimension of D_i is one, there is precisely one such fermion and the sheaf S^a is in fact the extension by zero of the line bundle $\mathcal{O}(q^a_{Z_i})_{D_i}$.

Note that this is the same argument that provides also the reason for why the basis $\{S^a\}$ is localized on the compact exceptional divisor X of the partial resolution $\mathcal{O}(-c_1(X))_X$ of $\mathbb{C}^{n+1}/\Gamma$. In this case, the relevant fermion that imposes the Dirichlet boundary condition for the resolution is the super-partner of the coordinate p on the fiber.

6. The McKay correspondence

In the previous section we constructed two finite bases of generators for $\mathcal{H}^{\rm top}_{\rm op}$, one from maps build from bosonic fields and the other from fermionic fields, and related them to sheaves in the geometric large volume phase. The intersection form of the two bases is determined by the group theoretical formulae (4.4) and (4.5) in the orbifold phase and coincides with the geometric formula (4.3) in the resolved phase. Moreover, by the non-degeneracy of the inner product χ_{ab}, they generate the closed string Hilbert space by their Chern classes if $\dim H^{\rm vert}(\tilde{X}) = N$. More precisely, the bases $\{R_a\}$ and $\{S^a\}$ are two bases of generators for the topological K-Theory group $K(\tilde{X})$, if $H^*(\tilde{X}) = H^{\rm vert}(\tilde{X})$. This is true for the case of toric manifolds that we consider [39].

A relation between the group theoretical data of the orbifold $\mathbb{C}^{n+1}/\Gamma$, and the homology of its resolution is a well-known subject in mathematics, the McKay correspondence. In particular Ito and Nakajima have introduced two bases for K-theory groups on $\mathbb{C}^3/\Gamma$ to formulate and prove a McKay correspondence in this case [23].[9]

From the previous considerations we see that *the McKay correspondence has a completely natural explanation in terms of the* $(2,2)$ *supersymmetric sigma model with boundaries.* As the connection with our study of the open string Hilbert space in the previous sections is rather clear, we will be brief in the following.

Claim. *The continuation of the bosonic and fermionic bases* $\{R_a\}$ *and* $\{S^{a\,*}\}$ *of* $\mathcal{H}^{\rm top}_{\rm op}$ *to large volume provides two orthogonal bases for the compact K-theory group* $K_c(\tilde{X})$ *on the linear sigma model resolution of* $\mathbb{C}^n/\Gamma$*. Moreover the basis* $\{R_a\}$ *extends to a basis of the K-theory on the non-compact space* $\mathbb{C}^{n+1}/\Gamma$.

[9]The importance of these bases for D-branes on the orbifold has been emphasized in [9, 14].

In particular, for $\mathbb{C}^3/\mathbb{Z}_N$, we identify the bases $\{R_a\}$ and $\{S^{a\,*}\}$ with the restriction of the constructions of Ito and Nakajima to the compact part $\tilde{X}$ of the resolution. This is clear from the fact that *i)* $\{R_a\}$ is a set of line bundles that generates $K(\tilde{X})$ *ii)* it provides a complete set of irreps of Γ (see below) *iii)* the basis $\{S^{a\,*}\}$ is orthogonal to $\{R_a\}$. Note that the open string point of view gives a literal identification between tensor products of Γ irreps and intersections on the resolution, in terms of the HRR identity (4.3) and its group theoretical form (4.4), (4.5). More specifically the interpolation via variation of FI terms gives also a continuation between the objects in the two different phases, LG fields in H representations in the orbifold phase and sheaves in the large volume phase, respectively.

A point that needs a little explanation is that the formula (4.5) involves representations of H and not of the discrete group Γ. However this issue is clear from the discussion in section 5.1: in fact Γ is the unbroken part of the continuous gauge group H in the small volume phase, and the tensor products of the H representations descend to those of Γ.

For this to be true it is of course necessary that the map from H to Γ reps is injective and onto irreps. In the present case of $H = \mathrm{U}(1)^r$ this is obvious, by construction. In fact we have reconstructed the bases on the resolution $\tilde{X}$ from the small volume basis $\{R_a\} = \{\mathcal{O}(n_0-N+1), \ldots, \mathcal{O}(n_0)\}$. The elements of $\{R_a\}$ define a complete set of irreducible representations ρ_a of $\Gamma = \mathbb{Z}_N$ by the identification

$$R_a \longrightarrow \rho_\alpha, \qquad \alpha = a \bmod N\,, \tag{6.1}$$

where ρ_α transforms under the $\mathbb{Z}_N$ with eigenvalue ω^α. Here ω is an N-th root of 1, $\omega^N = 1$.

For general gauge group H, the requirement that the map from H to Γ reps is injective and onto irreps may imply a non-trivial selection rule on H representations. This motivates the following conjecture:

Conjecture. *Consider a the set of generators $\{\tilde{S}^a\} \subset \mathcal{H}^{\mathrm{top}}_{\mathrm{op}}(\tilde{X})$ constructed as in the previous sections. Then there exists a subset $\{S^a\}$ of N elements in $\{\tilde{S}^a\}$ such that the two following conditions are equivalent:*

i) upon restriction from H to Γ, $\{S^a\}$ provides a complete set of irreps of Γ.

ii) the subset $\{S^a\}$ provides free generators of the derived category $D^\flat(\tilde{X})$.

7. Helices, mutations and the local mirror description

By construction, the two bases of generators $\{R_a\}$ and $\{S^a\}$ are defined in the orbifold limit as a set of N consequent elements of the infinite sets $\mathcal{H}_R$ and $\mathcal{H}_S$, respectively, as defined in section 4. We will now study this structure in more detail and identify the basis $\{R_a\}$ as the foundation of a helix $\mathcal{H}_R$ of exceptional sheaves, and similarly for $\{S^a\}$ and $\mathcal{H}_S$. In particular this will lead to a very effective definition of the basis $\{S^a\}$, in terms of short exact sequences involving the elements of the dual basis $\{R_a\}$.

We will also consider a different relation between sheaves with a large volume interpretation and states of a $(2,2)$ supersymmetric orbifold, namely local mirror symmetry. In a profound study of boundary states in $(2,2)$ supersymmetric two-dimensional QFT's [13], Hori, Iqbal and Vafa showed that helices of exceptional sheaves in a linear sigma model on a Fano variety X are related by local mirror symmetry to A-type boundary states of a Landau-Ginzburg theory.[10]The latter correspond to D-branes wrapped on lagrangian cycles in the mirror manifold.

In particular, the local mirror analysis of [13] was phrased in the mathematical framework of helices of exceptional sheaves. The link between helices and LG theory had been observed some time ago [40, 22] and its explanation has been one of the subjects of ref. [13]. Another relevant relation is the one between helices and quivers as an equivalence between derived categories [41].

7.1 A helix point of view

Let us sketch the definition of a helix of coherent sheaves; for more details we refer to [21, 22]. An *exceptional sheaf* E has $\mathrm{Ext}^0(E,E) = \mathbb{C}$ and $\mathrm{Ext}^k(E,E) = 0$ for $k > 0$. An *exceptional collection* of sheaves is an ordered collection of exceptional sheaves $\mathcal{E} = \{E_1, \ldots, E_N\}$ such that $\mathrm{Ext}^k(E_a, E_b) = 0$ for $a > b$ and for $a < b$ except at most for a single degree $k = k_0$. In particular, the index (4.3) has at most one non-trivial term equal to $(-1)^{k_0}\,\mathrm{Ext}^{k_0}(E_a, E_b)$. A *strong exceptional collection* has $k_0 = 0$ and may be used as a starting point to construct a map from the derived category of coherent sheaves E_n to the derived category of quiver algebras with relations [41].

The exceptional collection $\mathcal{E}$ has the important property that one may possibly define an operation, called *mutation* that acts on a pair in $\mathcal{E}$ and yields another exceptional set. One distinguishes left and right mutations that act as $\underline{L} : (E_{a-1}, E_a) \to (\underline{L}_{a-1}E_a, E_{a-1})$ and $\underline{R} : (E_a, E_{a+1}) \to (E_{a+1}, \underline{R}_{a+1}E_a)$, respectively. The Chern class of the mutated sheaves have a simple form, up to sign:

$$\begin{aligned} \mid\ \mathrm{ch}\,\underline{L}_{a-1}E_a\ \mid &= \mathrm{ch}\,E_a - \chi(E_{a-1}, E_a)\,\mathrm{ch}\,E_{a-1}\,, \\ \mid\ \mathrm{ch}\,\underline{R}_{a+1}E_a\ \mid &= \mathrm{ch}\,E_a - \chi(E_a, E_{a+1})\,\mathrm{ch}\,E_{a+1}\,, \end{aligned} \tag{7.1}$$

where the sign depends on the details (and represents the orientation in the context of D-branes [13]). Moreover the operations $\underline{L}$ and $\underline{R}$ satisfy the braid group relations. The definition of the operations $\underline{L}$ and $\underline{R}$ depends on details of the elements in $\mathcal{E}$. We refer to [21, 13] for a list of the definitions in the various cases.

[10]Although the discussion in [13] involves a certain class of examples, the arguments apply more generally and have a straightforward generalization to many other spaces.

Finally, the *helix* is defined as an infinite collection $\{E_a\}$ such that the set $\{E_{n_0+1}, \ldots, E_{n_0+N}\}$ is an exceptional collection for any n_0. Moreover $\underline{L}^{N-1} E_{n_0+N} = E_{n_0}$ and $\underline{R}^{N-1} E_{n_0+1} = E_{n_0+N+1}$, that is the $N-1$-th powers of $\underline{L}$ and $\underline{R}$ generate shifts of the origin by one to the left and to the right, respectively. In this way one may obtain the infinitely extended helix starting from an exceptional collection, which is then referred to as the foundation of the helix. The integer N is called the period of the helix.

We observe that the set $\mathcal{H}_R$ defined in section 4 obeys the criteria of the definition of a helix. Its period is defined by the first Chern class $c_1(\tilde{X})$ of the exceptional divisor. The basis $\mathcal{E}_R = \{R_a\} \subset \mathcal{H}_R$ of $K(\tilde{X})$ provides a foundation of the helix $\mathcal{H}_R$.

An interesting question is what the meaning of the mutations is in terms of the McKay bases $\{R^a\}$ and $\{S^a\}$. An evident operation is $\underline{L}^{N-1}$ ($\underline{R}^{N-1}$) which represents shift of the origin by one to the left (right) and leads to a foundation that generates the same helix. An arbitrary mutation leads in general to a foundation which generates a different helix. By (7.1) the classes of the new basis $\tilde{\mathcal{E}}$ present a linearly independent set of combinations of the classes in $\mathcal{E}$.

7.2 Mutations in the LSM and a generator for the helix $\mathcal{H}_R$

In [13], mutations have been identified as the monodromies in the moduli space of the local mirror LG model. However it is important to note that the moduli space of the LSM provides only an extremely restricted set of perturbations, and only a very small subset of mutations will be realized as monodromies. A canonical set of monodromies associated with the large volume limit of the LSM are the shifts $t_\alpha \to t_\alpha + 1$. They correspond to a shift of the B-field by K_α.

In section 9.3 we argue that there are always two distinguished monodromies, one, denoted by A, around the orbifold point and another, denoted by T, around a divisor in moduli space where a D6 brane shrinks to zero size. Moreover the inverse T_∞^{-1} of the combined monodromy $T_\infty = AT$ corresponds to the series of mutations

$$\{R_1, \ldots, R_N\} \longrightarrow \{R_2, \ldots, R_{N+1}\}, \tag{7.2}$$

and generates the helix $\mathcal{H}_R$ from the foundation $\{R_a\}$. Note that we may use the monodromy T_∞ to generate also the foundation $\{R_a\}$ from a single element, say R_1. In the orbifold phase, the analogue operation that generates the infinite set of ground states $\mathcal{H}_R$ is the multiplication (4.1) by a LG field of charge 1. Thus T_∞ is the generalization of this generating element to the resolved phase of the LSM. It is interesting to observe, that this monodromy is not an element of the large volume monodromy. It also does not preserve the distinguished large volume limit in general, see section 9.3 for an example. This is another reflection of the fact that the objects $\{R_a\}$ are well-defined throughout the moduli space, which makes the notion of a mutation meaningful even through the non-geometric regions in the moduli.

7.3 A reflection mutation $R_a \leftrightarrow S^{a\,*}$

We will now consider an important series of mutations that does not correspond to a monodromy of the linear sigma model. Its special feature is that it maps the dual bases $\{R_a\}$ and $\{S^{a\,*}\}$ onto each other. Specifically, we consider the operation $\underline{P}$ that acts as a reflection on a foundation $\mathcal{E}$:

$$\underline{P}: \ \mathcal{E} = \{E_1, E_2, \dots, E_N\} \longrightarrow \tilde{\mathcal{E}} = \{E_N, \underline{R}E_{N-1}, \dots, \underline{R}^{N-1}E_1\} . \tag{7.3}$$

From (7.1) we find that the Chern class of $\tilde{E}_a$ is

$$\tilde{E}_a = E_a + \sum_{b=a+1}^{N} c_{a,b} E_b \, , \tag{7.4}$$

where we write simply E_a for $\mathrm{ch}(E_a)$ and

$$c_{a,b} = -\chi_{ab} + \sum_{l=2}^{b-a} (-)^l \sum_{a<n_1<\cdots<n_{l-1}<b} \chi_{a,n_1}\,\chi_{n_1,n_2}\,\cdots\chi_{n_{l-1},b} \, . \tag{7.5}$$

These equations can be rewritten in compact form as

$$\tilde{E}_{N-a} = \left(\sum_{l=0}^{N} (-)^l \tilde{\chi}^l\right)^{ab} E_b = \left(\frac{1}{1+\tilde{\chi}}\right)^{ab} E_b = (\chi^{-1})^{ab}\, E_b \, , \tag{7.6}$$

where $\tilde{\chi} = \chi - 1$ satisfies $\tilde{\chi}^N = 0$. In particular, if we set $E_a = R_a$, we may identify

$$S^{a\,*} = \underline{R}^{N-a}\, R_a \, . \tag{7.7}$$

We see that the basis $\{S^{a\,*}\}$ of $K_c(X)$ represents the (dual of the) mutation $\underline{P}$ of $\{R^a\}$ of $K(X)$ and moreover $\tilde{\mathcal{E}}_S = \{S^1, \dots, S^N\} = \{S^{N\,*}, \dots, S^{1\,*}\}^*$ is the foundation of a helix.

7.4 A monodromy interpretation

In ref. [13] it was shown, that the exceptional bundles R_a are mapped to A-type boundary states of a LG model by local mirror symmetry. Moreover mutations of the exceptional collection have been identified with the monodromies of the LG model under variation of its deformation parameters. It is instructive to consider the monodromy corresponding to $\underline{P}$. This is best illustrated in the W-plane, the complex plane corresponding to the values of the superpotential in the mirror LG model (figure 2).

The five circles in figure 2 part a) correspond to the $\mathbb{Z}_5$ symmetric vacua w_a of the mirror LG model of $X = \mathbb{P}^4$. By a standard construction in singularity theory, one may define a complete basis of lagrangian 4-cycles[11] in the space of LG

[11] See [13, section 2] for a review of this construction.

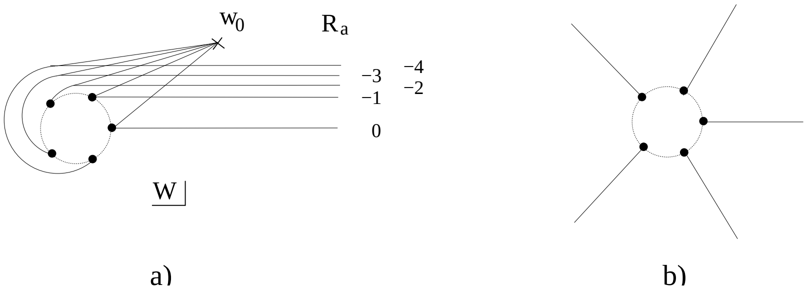

Figure 2: Mirror of the exceptional collection $\{R_a\}$ for $\mathbf{P}^4$.

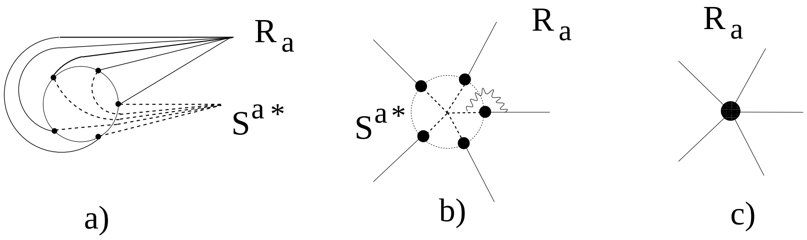

Figure 3: *The mirror of the two McKay bases: a)* the two McKay bases $\{R_a\}$ and $\{S^{a\,*}\}$ are related by the monodromy $\underline{P}$. *b)* the supersymmetric brane configurations converge to the points ∞ and $\mathbf{0}$, respectively. *c)* in the orbifold limit, the supersymmetric cycles in $\{S^{a\,*}\}$ collapse, while the non-compact basis $\{R_a\}$ survives.

fields by choosing a point p and considering its preimage $W^{-1}(p)$. This definition involves a choice of paths γ_a from p to the critical points w_a that defines a choice of basis; a change of the homotopy class of γ_a by moving it through a critical point w_b corresponds to a monodromy on the 4-cycles. By aligning the 4-cycles along the path γ_a and moving p to infinity, one may similarly define a basis of non-compact 5-cycles C_a spanning $H_5(\mathbb{C}^5, B)$, the homology of cycles with boundaries on the boundary $B: \ |W| = \text{const}$. Moreover the paths γ_a are identified in [13] as the image in the W-plane of the D-branes wrapped on the cycles C_a. The supersymmetric D-brane wrappings correspond to straight lines in the W planes with the slope determined by the phase in the definition of the preserved supercharge. This is shown in figure 2 part b).

The choice of path in figure 2 corresponds to the mirror image of the exceptional collection $\mathcal{E} = \{R_a\} = \{\mathcal{O}(-4), \ldots, \mathcal{O}\}$ [13]. As indicated, an ordering is defined by the vertical coordinate of a line to infinity, with an obvious modification if p is at a finite value.

A right (left) mutation of E_a corresponds to a monodromy in the W-plane, where the a-th line moves through the critical value next to it in (counter-)clockwise direction. The choice of path generated by the action of the reflection $\underline{P}$ is shown in figure 3 part a).

The bases of *supersymmetric* cycles homotopic to those in figure 3 part a) is shown in figure 3 part b). We see that the bases $\{R_a\}$ and $\{S^{a\,*}\}$ correspond to the two unique complete Γ symmetric bases of supersymmetric D-branes in the Landau-Ginzburg model. The orthogonality property of the two bases is evident, as the single possible massless ground state between the brane R_a and the brane $S^{b\,*}$ is an open string sitting at the critical point w_a.

In the orbifold limit, the critical points w_a move to the origin as shown in figure 3 part c). The basis R_a of non-compact cycles survives, while all the supersymmetric cycles in $\{S^{a\,*}\}$ collapse to a point.

7.5 A simple series of sequences for the basis $\{S^a\}$

The monodromy relation (7.7) between the bases $\{R_a\}$ and $\{S^{a\,*}\}$ leads to a simple construction of the sheaves S^a in terms of mutations. The definition of the mutation depends on the properties of the foundation $\mathcal{E}$; we refer to [41, 13, 22] for the definition in the various cases. We will write here the sequences for the dual basis $\{S^{a\,*}\}$ which is orthogonal to $\{R_a\}$. The sequences for S^a are simply the transpositions of those for $S^{a\,*}$. For the set $\{R_a\}$ a right mutation may be defined by the exact sequence [21]:

$$0 \longrightarrow E_a \xrightarrow{\alpha} \mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_{a+1} \xrightarrow{\beta} \underline{R}E_a \longrightarrow 0\,. \tag{7.8}$$

E.g. if the w_i are equal and thus $X = \mathbb{P}^n$, this is a twisted form of the well-known Euler sequence

$$0 \longrightarrow \mathcal{O}(k_a) \longrightarrow \mathbb{C}^{n+1} \otimes \mathcal{O}(k_a+1) \longrightarrow \Omega^*(k_a) \longrightarrow 0\,, \tag{7.9}$$

where $k_a = n_0 - N + a$ for the elements of the foundation $\{R_a\}$ with $R_N = \mathcal{O}(n_0)$. Similarly, from the relation $S^{a\,*} = \underline{R}^{N-a}\,R_a$, we may obtain the sequence for the sheaves $S^{a\,*}$ by repeated application of the mutation (7.8). It is simply the appropriate exterior product of the sequence (7.9), as can be easily seen by explicit construction of the maps. In this way we recover the result

$$S^{a\,*} = (-1)^{N-a}\,\Lambda^{a-1}\Omega(n_0+a)\,, \qquad a = 1, \ldots, N\,. \tag{7.10}$$

Specifically, eq. (7.10) is the dual of (5.6). Although the argument of multiplication by fermionic zero modes leads much more directly to this identification, the language of mutations provides a convenient closed form of the involved sequences for arbitrary weights w_i. On the other hand it is in fact not guaranteed that it is always possible to define the required series of mutations, as the elements of the mutated foundation may have Ext groups different from that in the original one. In this case, the two-dimensional point of view in section 5.4 provides a more general framework for the definition of the appropriate sequences.

8. D-branes on compact Calabi-Yau's and a McKay correspondence for singular resolutions

So far we have defined the sheaves R_a and S^a on the exceptional divisor X of a partial resolution of the singularity $\mathbb{C}^{n+1}/\Gamma$. We are now, as in [14], interested to study D-branes on a generic, compact Calabi-Yau Y, embedded as a hypersurface in X. This corresponds to the addition of a homogeneous superpotential $W \neq 0$ to the $(2,2)$ supersymmetric field theory (2.1).

The main result of this section is a simple relation between the data on X and Y that allows to go forth and back between the definition of the bases $\{R_a\}$ and $\{S^a\}$ on X and their restrictions to Y. This will allow us to define $\{S^a\}$ and $\{R_a\}$ directly on Y, even if the ambient space X is singular. This leads to a natural proposal for a McKay correspondence for quotient singularities that do not have a complete, crepant resolution. In particular there is still a correspondence between the group theoretical tensor product (4.5) and the intersections of the elements of the K-theory group on a smooth hypersurface Y in the maximal crepant resolution of $\mathbb{C}^{n+1}/\Gamma$ defined by the LSM.

8.1 The geometric setup from a string point of view

As we will focus from now on more concretely to D-branes in type-IIA compactification on Calabi-Yau manifolds, let us briefly summarize the geometric setup. The starting point is the by now familiar quotient singularity $Z = \mathbb{C}^{n+1}/\Gamma$, with $\Gamma \subset \mathrm{SU}(n+1)$ a discrete group such that there is a (partial) resolution $\tilde{Z}$ of Z with trivial canonical bundle. The space $\tilde{Z}$ may be relevant for string theory in two very different ways. Firstly, $\tilde{Z}$ may appear as a local patch of a Calabi-Yau $n+1$-manifold, on which the string propagates, e.g. a K3 manifold in the case $\mathbb{C}^2/\Gamma$. Secondly, we may embed a Calabi-Yau $(n-1)$-fold Y as a hypersurface (or intersections thereof) in the compact exceptional divisor $X \subset \tilde{Z}$. In this case the string propagates only on Y.

The two cases are closely related: to construct the Calabi-Yau Y we consider first the total space of the anti-canonical bundle $\mathcal{K}_X^*$. This is precisely the (partial) resolution of the non-compact Calabi-Yau Z. The hypersurface Y is then defined as the zero locus of a generic section of $\mathcal{K}_X^*$. E.g., for the quintic, $\tilde{Z}{=}\mathcal{O}(-5)_{\mathbb{P}^4}$ and Y is the zero locus of a quintic polynomial $p(x_i)$ in $X = \mathbb{P}^4$.

We may obtain elements of the K-theory group $K(Y)$ by restricting the large volume version of the bases $\{R_a\}$ and $\{S^a\}$ to the hypersurface [14]. This map is not bijective for small n but becomes better with increasing n [42]. A surprising consequence is that *the D-brane states on Y are to a large extent determined by the the representation theory of a simple orbifold*. We will describe in the next section how detailed information, such as monodromy matrices and analytic continuation of periods, may be extracted from the K-theory data defined in this way. Note that the orbifold limit $\mathrm{Vol}(X) \to 0$ implies also a small volume limit of the hypersurface Y.

8.2 A McKay correspondence for singular resolutions

The question inverse to the restriction of $K(X)$ to $K(Y)$ is: *to what extent do the K-theory data on Y describe a McKay correspondence* ? This becomes important in dimensions higher than three, as not all the singularities of the form $Z = \mathbb{C}^{n+1}/\Gamma$

with $\Gamma \subset \mathrm{SU}(n+1)$ allow for a resolution that keeps the canonical bundle trivial and a generalization of the McKay correspondence is not obvious. However, a Calabi-Yau hypersurface embedded in a partial resolution of Z may avoid the remaining singularities. We will argue now that the bases $\{R_a\}$ and $\{S^a\}$ can be specified entirely on the hypersurface Y embedded in the partial resolution defined by the linear sigma model. For $n > 2$ this gives a concrete proposal for a McKay correspondence, even if the maximal, crepant resolution of Z is still singular: we may define the McKay correspondence as a relation between the group theory of $\mathbb{C}^{n+1}/\Gamma$ and the compact K-theory group $K_c(H)$ generated by $\{S^a\}$ on a smooth, compact hypersurface H of minimal codimension in the maximal, crepant resolution $\tilde{Z}$ of Z as defined by the LSM.

Let us consider again the inner product (4.3). The Todd classes of X and Y are related by

$$\mathrm{td}(X) = \frac{1}{2}\, c_1(X)\ \mathrm{td}(Y) + \text{even}\,, \tag{8.1}$$

where the term (even) denotes that it contains only $(2k, 2k)$-forms. From $\int_Y (.) = \int_X c_1(X)(.)$, we have

$$\begin{aligned} \langle A, B\rangle_X &= \frac{1}{2}\,\langle A, B\rangle_Y + X(A, B)\,, \\ \langle B, A\rangle_X &= \frac{1}{2}\,\langle A, B\rangle_Y\,(-)^{n-1} + X(A, B)\,(-)^n\,, \end{aligned} \tag{8.2}$$

where $n = \dim_{\mathbb{C}}(X)$ and the form of $X(A, B)$ is irrelevant up to the fact that it transforms with a sign $(-)^n$ under the exchange of A and B. Note also that $\mathrm{td}(Y)$ contains only $(2k, 2k)$-forms, which implies that $\langle A, B\rangle_Y$ transforms with a sign $(-)^{n-1}$ under the same exchange. It follows that

$$\langle A, B\rangle_Y = \langle A, B\rangle_X + (-)^{n-1}\,\langle A, B\rangle_X\,. \tag{8.3}$$

If $\langle A, B\rangle_X$ is upper triangular, as is the case for A, B are elements of $\{R_a\}$ or $\{S^a\}$, it is completely determined in terms of $\langle A, B\rangle_Y$:

$$\chi_{ab} = \langle R_a, R_b\rangle_X = \begin{cases} \langle R_a|_Y, R_b|_Y\rangle_Y\,, & b > a \\ 1\,, & b = a \\ 0\,, & b < a\,. \end{cases} \tag{8.4}$$

The formulae (8.3) and (8.4) allow to go forth and back between the data on Y and that on X. Note that the inner product $\langle R_a, R_b\rangle_Y$ is still completely determined by the representation theory of Γ; it is not invertible on Y, however. The restrictions $V^a = S^a|_Y$ in terms of the data on Y are

$$V^a = \chi^{ab} R_b^*|_Y\,, \tag{8.5}$$

with χ^{ab} the inverse of the matrix defined in (8.4). The intersections of the V^a on Y are related to that on X by (8.3), $\langle V^a, V^b\rangle_Y = \chi^{ab} + (-)^{n-1}\chi^{ba}$. This completes the relation between the representation theory of Γ and the K-theory group with compact support on Y. We refer to appendix C for a simple example, where the LSM does not provide a smooth resolution and thus the ambient space X is singular.

9. Application to Calabi-Yau three-folds

In this section we apply the previous ideas to study stringy D-branes on a Calabi-Yau 3-fold Y. The investigation of this subject was initiated in [8] for the quintic hypersurface in $\mathbb{P}^4$, falling back on previous work on boundary states in CFT [7, 27]. See also [43]–[45], [12, 25, 26, 46] for subsequent studies of this subject.

The main focus of this section is to demonstrate that the representation theory of the embedding singularity contains a surprising amount of information about the closed string sector on the compact Calabi-Yau manifold embedded in it. The main ingredient is the map between D-branes at small and large volume described in the previous sections which, apart from being much simpler then the approaches used so far, allows to reconstruct detailed data of the closed string sector, namely intersections, the large volume prepotential, monodromies and analytic continuation matrices, from the simple group theoretical data of the open string sector.

9.1 Fractional branes

The natural, fundamental objects of the type-II string at small volume are the D-branes wrapped on the compact cohomology of X, the so-called fractional branes. They correspond to the generators S^a of the basis of $K_c(X)$. In [14] it was conjectured, that the restrictions of the fractional brane states to a Calabi-Yau Y embedded as a hypersurface in X, represent the rational B-type boundary states of a Gepner model that describes the small volume limit of Y. Let us give an explicit expression for the intersection form of the restriction $V^a = S^a|_Y$ of the fractional branes to a 3-fold hypersurface Y. From (8.3) we obtain:

$$\begin{aligned} I^{ab}_{LG} &= \langle V^a, V^b\rangle_Y = \chi^{ab} - \chi^{ba} = \\ &= \Big(\prod_i (1-h^{w_i}) - \prod_i (1-h^{w_i})^T\Big)^{ab} = \Big(\prod_i (1-g^{w_i})\Big)^{ab}. \end{aligned} \qquad (9.1)$$

Here g is the extended shift matrix $g = h + \delta_{N,1}$. The notation indicates that this intersection form is to be identified with that of the LG states, according to the conjecture. This quantity may be calculated if the theory has a representation as a Gepner model [8]. The result is precisely the last expression in (9.1). This proves that the intersection matrices of the fractional branes and the LG boundary states is the same and gives strong evidence for their identification. In fact the result (9.1) does not depend on the existence of a Gepner model and generalizes to all hypersurfaces in weighted projective space.

9.2 An open string derivation of the large volume prepotential

The integral Chern classes of the fractional branes may be determined from (8.5) by simple linear algebra. The other preferred, integral lattice that labels the charges of the D-brane states is the symplectic lattice of BPS charges in the closed string theory discussed in section 5. The classes of the fractional branes on the 3-fold Y and their integral, symplectic charges $\vec{Q}$ are related by

$$Z(A) = -\int_Y e^{-J}\ \mathrm{ch}(A)\ \sqrt{\mathrm{td}(Y)} = \vec{Q}\cdot\vec{\Pi}\,, \tag{9.2}$$

where $J = \sum_{\alpha=1}^{h_{1,1}} t_\alpha\, K_\alpha$ is the Kählerform on Y and A an element of the K(Y). The expression on the left hand side[12] is derived from the coupling of the D-brane world volume theory to the to the background fields [33] and the right hand side is the well-known central charge formula in the closed string theory [47]. Equating the central charge of the D-brane states as obtained from the open or closed string picture, respectively, leads to the above relation [8, 32].

The symplectic basis is specified by the 'period vector' $\vec{\Pi}$ which is the section of a $SL(2h_{1,1}+2,\mathbb{Z})$ bundle. Its general form is determined by $\mathcal{N}=2$ space-time supersymmetry in terms of a holomorphic prepotential $\mathcal{F}(t_\alpha)$:

$$\vec{\Pi}(t_\alpha) = \begin{pmatrix} \Pi_6 \\ \Pi_4^\alpha \\ \Pi_0 \\ \Pi_2^\alpha \end{pmatrix} = \begin{pmatrix} 2\mathcal{F} - t_\alpha\partial_\alpha\mathcal{F} \\ \partial_\alpha\mathcal{F} \\ 1 \\ t_\alpha \end{pmatrix}. \tag{9.3}$$

It is easy to see that the large volume form of the prepotential $\mathcal{F}$ is determined by the open string formula (9.2) and we need not to invoke mirror symmetry to determine it.[13] The leading terms of the central charge Z for a 6-brane and a 4-brane that is wrapped on the divisor E_α dual to K_α are

$$Z_6 = \int_Y \frac{J^3}{3!} + J\,\frac{c_2(Y)}{24}, \qquad Z_4 = \int_Y \frac{-J^2\,K_\alpha}{2} + \frac{J}{2}\,i_* c_1(E_\alpha). \tag{9.4}$$

Here i denotes the embedding $i : E \hookrightarrow Y$ and we have used the Grothendieck-Riemann-Roch formula to relate the K-theory class on Y and the Mukai vector of the bundle on E. It follows that the large volume form of the prepotential is related to the topological intersection data on Y by

$$\mathcal{F} = -\frac{1}{3!}\,C_{\alpha\beta\gamma}t_\alpha t_\beta t_\gamma + \frac{1}{2!}t_\alpha t_\beta A_{\alpha\beta} + B_\alpha t_\alpha\,, \tag{9.5}$$

[12] This K-theoretic expression and the more conventional form involving the Mukai vector of a bundle are related by a Grothendieck-Riemann-Roch argument, as explained in [32].

[13] See [48, 49] and references therein for the derivation of $\mathcal{F}$ from mirror symmetry and [50] for a related discussion.

with

$$
\begin{aligned}
C_{\alpha\beta\gamma} &= \int_Y K_\alpha K_\beta K_\gamma \,, \\
B_\alpha &= \frac{1}{24}\int_Y K_\alpha\, c_2(Y) \bmod \mathbb{Z}\,, \\
A_{\alpha\beta} &= \frac{1}{2}\int_Y K_\beta\, i_* c_1(E_\alpha) \bmod \mathbb{Z}\,.
\end{aligned} \tag{9.6}
$$

A simple check of these expressions is given by the requirement that the period vector $\vec{\Pi}$ transforms by a symplectic transformation under the shifts of the B-field, $t_\alpha \to t_\alpha+1$. This implies $A_{\alpha\beta}+1/2C_{\alpha\alpha\beta} \in \mathbb{Z}$, $2B_\alpha+1/6C_{\alpha\alpha\alpha} \in \mathbb{Z}$ which is guaranteed by the relations $2B_\alpha = -1/6C_{\alpha\alpha\alpha} + \chi(E_\alpha) \mod \mathbb{Z}$ and $A_{\alpha\beta} = -1/2C_{\alpha\alpha\beta} \mod \mathbb{Z}$.

From eqs. (9.3) and (9.2) we may then obtain the relation between the Chern classes and the closed string charges $\vec{Q}$ for any brane configuration.

9.3 From open to closed strings: a representative example

In the following we will use the information from the open string sector, namely the fractional branes, to obtain detailed information on the moduli space of the closed string sector on Y. As a basic example note that we can determine the intersection data (9.6) on Y from the tensor product formula (4.4) by expressing the Kähler classes K_α in terms of Chern classes of the generators R_a of the K-theory group.

It seems useful to explain the following arguments on the basis of a concrete example, which we choose to be the degree 24 hypersurface embedded in a partial resolution of a $\mathbb{C}^5/\mathbb{Z}_{24}$ singularity. The Calabi-Yau manifold Y has $h_{1,1}=3$ and has a sufficient degree of complexity to serve as a representative example in many respects. Firstly the smooth hypersurface Y may be embedded in a singular ambient space and provides an example for a McKay correspondence in a singular resolution.[14] Also, while it would be hard to approach with the closed string methods, namely analytic continuation used in [8, 43, 32, 44, 45, 12] or the toric description of refs. [10], [14], it is easy to deal with from the point of the open string picture and thus demonstrates the effectiveness of this framework.

9.3.1 The group theoretical input

The discrete group is defined by the weight vector $w=(1,1,2,8,12)$ and acts as $x_i \to \omega^{w_i} x_i$ on the coordinates, with $\omega^{24}=1$. A partial resolution of this space is provided by the total space of the bundle $\mathcal{O}(-24)_X$, $X = \mathbb{WP}^4_{1,1,2,8,12}$. The Calabi-Yau hypersurface Y is defined as the zero locus of a generic section of this bundle.

[14] In this case, however, the additional singularities *do* have a crepant resolution in the linear sigma model, so that the smooth hypersurface of minimal codimension in X is the ambient space X itself. This will lead to a trivial linear dependence of the basis $\{S^a\}$ in $K(Y)$, as further discussed below.

The starting point for the construction of the fractional branes are the group theoretic formula (4.4), (4.5) for the inner product of the elements in $\{R_a\}$, and its inverse that describes classes the fractional branes $\{S^a\}$ and their intersections:

$$\chi^{\mathbb{Z}_{24}}_{ab}=\begin{pmatrix} 1 & 2 & 4 & 6 & 9 & 12 & 16 & 20 & 26 & 32 & 40 & 48 & 59 & 70 & 84 & 98 & 116 & 134 & 156 & 178 & 205 & 232 & 264 & 296 \\ 0 & 1 & 2 & 4 & 6 & 9 & 12 & 16 & 20 & 26 & 32 & 40 & 48 & 59 & 70 & 84 & 98 & 116 & 134 & 156 & 178 & 205 & 232 & 264 \\ & & & & & \vdots & & & & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\chi^{\mathbb{Z}_{24}\,ab} = \begin{pmatrix} 1 & -2 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 1 & 0 & 0 & 0 & 1 & -2 & 0 & 2 \\ 0 & 1 & -2 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 1 & 0 & 0 & 0 & 1 & -2 & 0 \\ & & & & & \vdots & & & & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (9.7)$$

Note that the k-th row is just a shift of the first row by $k-1$ to the right, with zeros appended at the left. The matrix χ^{ab} specifies the classes of the fractional branes S^a in terms of that of the R_a by (4.6).

We may choose the elements R_a in the foundation $\{R_a\}$ of the helix $\mathcal{H}_R$ to lie at the origin, $\{R_a\} = \{\mathcal{O}(-23\,K), \dots, \mathcal{O}\}$. Here K denotes the hypersurface class of X.

9.3.2 The toric resolution

To proceed we perform a toric resolution of the singularities of the weighted projective space. A complete resolution is described by the fan spanned by the following vertices

$$\begin{gathered}(0,0,0,-1),(0,0,-1,0),(0,0,2,3),(0,-1,0,0),(0,1,4,6),(-1,0,0,0),(1,2,8,12),\\ (0,0,0,1),(0,0,1,1),(0,0,1,2).\end{gathered} \qquad (9.8)$$

However only the divisors associated to the vertices in the first line do intersect the generic hypersurface [38]. As we have argued, by the relation (8.5), we may define the bases $\{R_a\}$ and $\{S^a\}$ entirely on Y and their intersections in terms of $\langle A, B\rangle_Y = \int_Y \mathrm{ch}\,A^* \ \mathrm{ch}\,B\ \mathrm{Td}(Y)$. Therefore we will work on the singular space $\hat{X}$ with the singularities corresponding to the last three vertices in (9.8) unresolved.

In the large volume phase, the gauged LSM has the gauge symmetry $\mathrm{U}(1)^{h_{1,1}=3}$ and matter fields with charges

$$\begin{array}{c|cccccccc|c} & p & x_5 & x_4 & z & x_3 & w & x_2 & x_1 & \\ \hline l_1 & -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & \times 4 \\ l_2 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & \times 2 \\ l_3 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & \times 1 \\ \hline l_K & -24 & 12 & 8 & 0 & 2 & 0 & 1 & 1 & \end{array} \qquad (9.9)$$

The vectors l_α span the Kähler cone of Y in the large volume phase. We have also indicated the single U(1) charge vector $l_K = 4\,l_1 + 2\,l_2 + l_3$ of the fields in the LG phase.

It will be useful to have a little understanding of the geometry of the manifold Y. The fields $x_i,\ i = 1, \dots, 5$ are the original coordinates on the weighted projective space $\mathbb{WP}^4_{1,1,2,8,12}$. The fields z and w are the extra fields introduced in the resolution and associated to the exceptional divisors. The manifold Y is an elliptic fibration over the Hirzebruch surface $\mathbf{F}_2$. The elliptic fiber E is parametrized by the coordinates (x_4, x_5, z), while the Hirzebruch surface is the divisor $\mathbf{F}_2 : z = 0$, parametrized by (x_1, x_2, x_3, w). The base $\mathbf{F}_2$ is itself a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$, where the former, denoted by F, is parametrized by (x_3, w) and the latter is the divisor $B : w = 0$ in $\mathbf{F}_2$ with coordinates (x_1, x_2).

9.3.3 The basis $\{R_a\}$ of geometric bundles

We have already specified the elements of $\{R_a\}$ in the LG phase. From the $\mathrm{U}(1)^3$ charges of the fields X_i and the relation of the charge vectors l_α to l_K, we may reconstruct the classes of line bundles $\hat{R}_a$ on the partial resolution $\hat{X}$ as explained in section 5.3:

$$\begin{aligned}\mathcal{E}_{\hat{X}} &= \{\mathcal{O}(-5,-1,-1), \mathcal{O}(-5,-1,0), \mathcal{O}(-5,0,-1), \mathcal{O}(-5,0,0), \mathcal{O}(-4,-1,-1), \ldots \\ &\quad \ldots, \mathcal{O}(-1,0,-1), \mathcal{O}(-1,0,0), \mathcal{O}(0,-1,-1), \mathcal{O}(0,-1,0), \mathcal{O}(0,0,-1), \mathcal{O}(0,0,0)\} \\ &\stackrel{LG}{\longrightarrow} \{\mathcal{O}(-23), \mathcal{O}(-22), \ldots, \mathcal{O}\}. \end{aligned} \tag{9.10}$$

The bundles $\hat{R}_a$ in $\mathcal{E}_{\hat{X}}$ descend to the ground states R_a in the LG phase, as indicated. Here we use the standard notation $[\mathcal{O}(a)] = a\,K$ and $[\mathcal{O}(a,b,c)] = a\,K_1 + b\,K_2 + c\,K_3$, where K_α are the $(1,1)$ forms on $\hat{X}$ related to the generators l_α.

We could perform a further resolution of $\hat{X}$ to $\tilde{X}$ to blow up the remaining singularities in the ambient space. We may reconstruct the basis $\{\tilde{R}_a\}$ on $\tilde{X}$ in the same way as above from the charges of the fields w.r.t. to the $\mathrm{U}(1)^6$ gauge symmetry in this phase. This realizes our claim that we can go from the data on the resolutions to that on the singular space and *vice versa*. However there is no purpose for us to do so and we will continue to work with the partial resolution $\hat{X}$ that resolves only the singularities on the hypersurface.

9.3.4 The fractional branes V^a

We will now describe the structure of the fractional branes V^a on Y and find complete agreement with the arguments in section 2. The orbit of 24 fractional branes V^a comes in two sets of 12, which describe the branes and anti-branes on Y. The reason for the simple linear dependence of the charges V^a with $a > 12$ on those with $a < 12$ is the fact that Y misses singularities of the ambient space X which may be resolved. In the resolution, there are new K-theory classes on $\tilde{X}$ which lie in the kernel of the map $K(\tilde{X}) \to K(Y)$. Note that the same will not happen if Y misses singularities that do *not* have a crepant resolution.

Let us first consider the 12 branes. With $i:\ \mathbf{F}_2\to Y$ denoting the inclusion, the 12 branes build on $V^1=\mathcal{O}_{(-1,1,1)}$ correspond to the sheaves $V^a=\mathcal{O}_{(-1,1,1)}\otimes\tilde V^a$ with

$$\begin{array}{lll}
\tilde V^1=+\mathcal{O}_{(0,0,0)}\,, & \tilde V^5=-i_*\mathcal{O}_{(0,-2,0)}\,, & \tilde V^9=-\mathcal{O}_{(1,-2,0)}\,,\\
\tilde V^2=-\mathcal{O}_{(0,0,1)}\,, & \tilde V^6=+i_*\mathcal{O}_{(0,-2,1)}\,, & \tilde V^{10}=+\mathcal{O}_{(1,-2,1)}\,,\\
\tilde V^3=-\mathcal{O}_{(0,1,-2)}\,, & \tilde V^7=+i_*\mathcal{O}_{(0,-1,-2)}\,, & \tilde V^{11}=+\mathcal{O}_{(1,-1,-2)}\,,\\
\tilde V^4=\mathcal{O}_{(0,1,-1)}\,, & \tilde V^8=-i_*\mathcal{O}_{(0,-1,-1)}\,, & \tilde V^{12}=-\mathcal{O}_{(1,-1,-1)}\,.
\end{array}\tag{9.11}$$

Note that the sign in the definitions (9.11) counts precisely the fermion number of a map $V^1\to V^a$.

To interpret the sheaves V^a in the spirit of the previous sections, note first, that the fermions $\psi^1,\dots,\psi^5$ live on the whole space $Y\subset X=\mathbb{WP}^4_{1,1,2,8,12}$. As we discussed already, this a consequence of the fact that the associated super-fields parametrize the exceptional divisor X of the first resolution of the orbifold $\mathbb{C}/\mathbb{Z}_{24}$. On the other hand, the fermions ζ^z and ζ^w are related to resolutions of the singularities on $Y\subset X$. They introduce new boundary conditions that correspond to branes wrapped on the divisors $z=0$ and $w=0$, respectively.

The first non-trivial bundle V^2 is obtained in the LG phase by multiplication of $\mathcal{O}$ with $\psi^{1,2}$. The corresponding bundle is described by the restriction of the sequence (5.5) for the tangent bundle on X to B. We may thus identify V^2 with the pull back of the tangent bundle $\Omega^*_B(-K_3)$ to Y. Note that $\{V^1,V^2\}$ is the pull back of the helix $\mathcal{H}_{\mathbb{P}^1}=\{\mathcal{O},\Omega^*_{\mathbb{P}^1}(-1)\}$ on $\mathbb{P}^1$ to Y.[15]

The bundle V^3 is obtained in the LG phase by multiplication of $\mathcal{O}$ by ψ^3. In the geometric phase, this field gets part of the tangent bundle of the sphere F, with sections ψ^3,ζ^w. As the field w is also a section of $\mathcal{O}(-2\,K_3)$, the resulting bundle is the pull back of $\Omega^*_F(-K_2-2K_3)$ to Y, in agreement with (9.11). Similar the bundle V^4 corresponds to multiplication of $\mathcal{O}$ by $\psi^3\psi^1$. From the above, we identify V^4 as the pull back of $\Omega^*_B\otimes\Omega^*_F(-K_2-3K_3)$.

The next set of four elements is more interesting as they correspond to sheaves on $\mathbf{F}_2$ extended by zero. Note that this is precisely the situation where the map from $V^1\to V^5$ is gauge equivalent to zero and moreover there is no map $V^1\to V^a$ in the LG phase for $a=6,7,8$. However, in the geometric phase, there is the extra field ζ^z which generates non-trivial sections of these sheaves. Multiplication by ζ^z sets to zero the bosonic component z and thus the new brane sits on the divisor $\mathbf{F}_2:\ z=0$. This agrees perfectly with the result in (9.11)! We thus identify V^5 with $i_*\mathcal{O}_{\mathbf{F}_2(0,-2,0)}$; the next three sheaves are constructed precisely as in the first set of four above and correspond to the pull backs of the bundles $\Omega^*_B(-2K_2-K_3)$, $\Omega^*_F(-3K_2-2K_3)$ and $\Omega^*_B\otimes\Omega^*_F(-3(K_2+K_3))$ to $\mathbf{F}_2$, further extended by zero on Y.

As for the next set of four, there is again a fermion, namely ψ^4, that maps $V^1\to V^9$ in the LG phase and yields thus again a bundle on Y. Note that the

[15] As the dimension of B is one, we obtain only line bundles. See appendix B for a completely analogue example in a less degenerate case.

bundle V^a is defined as the restriction of S^a to Y and thus V^9 is determined by the section associated with ψ^4 on X, not on Y.

Finally the set of 9 anti-branes is constructed in the very same way as the set of the 9 branes with the difference that there is one extra factor ψ^5 in the definition of their sections, which gives the minus sign for each V^a. We note also that the repetition pattern in groups of four is related to the shift of the origin of the dual foundation $\{R_a\}$ by four units. In particular we will show below that this shift corresponds to the monodromy $t_1 \to t_1 + 1$ which preserves the large volume limit in the Calabi-Yau moduli.

9.3.5 The BPS charge lattice

It is straightforward to determine the topological data (9.6) of Y. They are summarized by the following prepotential:

$$\mathcal{F} = -t_3t_2t_1 - \frac{4}{3}t_1^3 - t_3t_1^2 - t_2^2t_1 - 2t_1^2t_2 + \frac{23}{6}t_1 + 2t_2 + t_3 \,, \tag{9.12}$$

up to terms constant and exponential in the t_α. From (9.2) we obtain the following $N \times 2h_{1,1}+2$ matrix Q_S which describes the symplectic charges $\vec{Q}$ for the fractional branes $V^a = S^a|_Y$:

	Q_6	Q_4^1	Q_4^2	Q_4^3	Q_0	Q_2^1	Q_2^2	Q_2^3
V^1	-1	1	-1	-1	-2	0	1	0
V^2	1	-1	1	2	0	-1	-2	0
V^3	1	-1	2	-1	2	0	-1	-1
V^4	-1	1	-2	0	-1	0	2	1
V^5	0	1	-2	0	0	0	1	0
V^6	0	-1	2	0	0	0	-2	0
V^7	0	-1	2	0	0	0	-1	-1
V^8	0	1	-2	0	-1	0	2	1
V^9	1	0	-1	1	2	0	0	0
V^{10}	-1	0	1	-2	0	1	0	0
V^{11}	-1	0	0	1	-2	0	0	0
V^{12}	1	0	0	0	0	0	0	0

$$\vec{Q}(V^{12+a}) = -\vec{Q}(V^a)\,. \tag{9.13}$$

9.3.6 Monodromies

From the symplectic charges of the fractional branes we may derive further information about the monodromies of the Kähler moduli space $\mathcal{M}_K$ of Y. Let us first consider the monodromy around the divisor C_0 in $\mathcal{M}_K$ which corresponds to the LG point at small volume.[16] At this point, the moduli space has a $\mathbb{Z}_N$ monodromy that

[16]To be precise, $\mathcal{M}_K$ denotes enlarged Kähler moduli space which has been resolved such that the components of the discriminant intersect with normal crossings.

permutes cyclically the line bundles of a given foundation, $R_a \to R_{a-1}$ for $a > 1$ and $R_1 \to R_N$. From the orthogonality relation between the R_a and the S^a it follows, that the effect on the S^a is the same. This transformation may be realized as a left multiplication of Q_S by the matrix g^T. The fact that this transformation corresponds to a monodromy in the moduli space implies the existence of a $(2h_{1,1}+2, 2h_{1,1}+2)$ matrix A that fulfills $g^T \cdot Q_S = Q_S \cdot A$. It is straightforward to determine the matrix A from (9.13):

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 & 1 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & -1 & 1 & 2 & 1 \\ 1 & -2 & 4 & 0 & 1 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}. \tag{9.14}$$

We may also consider the action on Q_S under a shift of the origin of the foundation $\{R_a\}$ by one to the left. From the previous discussion we know that a shift by N represents multiplication of the basis $\{R_a\}$ by $c_1(M)$. However we do expect that even a single shift represents a monodromy around a component C_∞ of the discriminant locus, as the origin of the foundation $\{R_a\}$ should not be distinguished. Indeed we find that the action $\{R_{n_0+1}, \ldots, R_{n_0+N}\} \to \{R_{n_0}, \ldots, R_{n_0+N-1}\}$ has a representation by right multiplication of Q_S with a matrix $T_\infty^{-1} = (AT)^{-1}$, where T is the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.15}$$

We recognize T as the monodromy of the conifold point, where the 6-cycle corresponding to the whole space Y shrinks to zero size. We have checked the above relation between the unit shift of the origin of $\{R_a\}$ and the monodromy AT, with T the monodromy of the zero size 6-brane in many other examples which suggests that there should be a general reason for this universal behavior.

Let us also consider the action of T_∞ on the Kähler coordinates t_α. Whereas the monodromy AT does not commute with the distinguished large volume limit, its fourth power does:

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \overset{AT}{\to} \begin{pmatrix} t_3t_2 + t_2^2 - t_1 - 2t_2 - t_3 + 7/6 \\ -2t_3t_2 - 2t_2^2 + 3t_2 + t_3 - 4/3 \\ -1 + t_3 \end{pmatrix} \overset{AT}{\to} \begin{pmatrix} -1 + t_1 + t_2 \\ 1 - t_2 \\ t_3 \end{pmatrix} \overset{(AT)^2}{\to} \begin{pmatrix} -1 + t_1 \\ t_2 \\ t_3 \end{pmatrix}. \tag{9.16}$$

In particular we see that a shift by $N = 24$ corresponds to a shift of the B-field by $c_1(M) = 6\,K_1$, in full agreement with the results obtained in [13] from local mirror symmetry. We note also that the monodromy action $(AT)^2$ induces a $\mathbb{Z}_2$ symmetry of the instanton expansion of Y, as it preserves a large volume limit but acts non-trivially on the t_α.

9.3.7 Analytic continuation from the Gepner point to large volume

A major obstacle to reconstruct the global structure of the moduli space $\mathcal{M}_K$ is to determine the precise linear combination of local solutions to differential equations satisfied by the periods that define a symplectic section $\vec{\Pi}$. One way to find this relation, used first in [51] for the quintic with $h_{1,1} = 1$, is the analytic continuation of period integrals. It is very hard to generalize this technique to the case of larger $h_{1,1}$. The open string approach gives an effective way to determine the analytic continuation from the LG point to large volume by simple linear algebra.

At the LG point, the N D-brane states V^a come in an orbit of the $\mathbb{Z}_N$ symmetry with degenerate masses $\sim |Z(V^a)|$. One may use $2h^{1,1}+2$ of the central charges $\omega_a = Z(V^a)$ as a basis for the period vector $\vec{\Pi}$ at the LG point and we may choose $\vec{\Pi}_G = (\omega_0, \ldots, \omega_{2h_{1,1}+2})$. Note that this is not yet a symplectic section of $SL(2h_{1,1}+2,\mathbb{Z})$, but related to it by a linear transformation. The remaining periods ω_a may be expressed in terms of those in $\vec{\Pi}$. One way to determine these relations is to study explicit expressions of period integrals as in [51]. However there is a simpler way, given the intersection matrix $I_{LG}^{ab} = \chi^{ab} - \chi^{ba}$: the relations correspond simply to its zero vectors. There are $N - (2h_{1,1}+2)$ of them and they take the following form in the present example:

$$\begin{aligned}
\omega_9 &= -\omega_1 + \omega_5\,, & \omega_{10} &= -\omega_2 + \omega_6\,, & \omega_{11} &= -\omega_3 + \omega_7\,,\\
\omega_{12} &= -\omega_4 + \omega_8\,, & \omega_{13} &= -\omega_1\,, & \omega_{14} &= -\omega_2\,,\\
\omega_{15} &= -\omega_3\,, & \omega_{16} &= -\omega_4\,, & \omega_{17} &= -\omega_5\,,\\
\omega_{18} &= -\omega_6\,, & \omega_{19} &= -\omega_7\,, & \omega_{20} &= -\omega_8\,,\\
\omega_{21} &= \omega_1 - \omega_5\,, & \omega_{22} &= \omega_2 - \omega_6 & \omega_{23} &= \omega_3 - \omega_7\,,\\
\omega_{24} &= \omega_4 - \omega_8\,.
\end{aligned} \tag{9.17}$$

We may now determine the analytic continuation matrix M between the central charges $Z(V^a)$ and the large volume periods $\vec{\Pi}$ by the formula

$$Q_S \cdot M = (1-g) \cdot \begin{pmatrix} \mathbf{1}_{2h_{1,1}+2} \\ R \end{pmatrix}, \tag{9.18}$$

where R is the matrix of relations in (9.17). The factor $(1-g)$ takes into account the fact that the LG states correspond to semi-periods, rather than periods of the LG model, see ref. [52] and section 5.2 of ref. [13]. In this way we find the searched for analytic continuation matrix

$$M = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & -1 & -1 \\ 2 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix}. \tag{9.19}$$

Note added. On the date of publication we received the preprints [53] which discuss similar issues and have a certain overlap with the results in section 7 and section 9.[17]

Acknowledgments

I am very grateful to Mike Douglas and Wolfgang Lerche for important remarks. I would also like to thank Emanuel Diaconescu, Sheldon Katz, Andy Lutken, Yaron Oz, Christian Römelsberger, Christof Schmidhuber and Johannes Walcher for valuable discussions and the Rutgers physics department for hospitality during completion of this work.

A. A simple example

Let us exemplify the above construction on the basis of the simple example $\mathbb{C}^5/\mathbb{Z}_5$ with $\mathcal{O}(-5)_{\mathbb{P}^4}$ as the partial and complete resolution. The $\mathbb{Z}_5 \subset U(1) \subset \mathbb{C}^*$ acts on the homogeneous coordinates of $\mathbb{P}^4$ as $x_i \to \omega x_i$, with $\omega^5 = 1$. The irreps ρ_a of $\mathbb{Z}_5$ transform as ω^a, $a = 0, \ldots, 4$ There are 5 linear monomials generating the maps from ρ_a to ρ_{a+1}, 15 quadratic monomials generating the maps ρ_a to ρ_{a+1}, and so on. The bilinear form χ^H_{ab} and its inverse, obtained from (4.4), (4.5), are

$$\chi^{\mathbb{Z}_5}_{ab} = \begin{pmatrix} 1 & 5 & 15 & 35 & 70 \\ 0 & 1 & 5 & 15 & 35 \\ 0 & 0 & 1 & 5 & 15 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \chi^{\mathbb{Z}_5\, ab} = \begin{pmatrix} 1 & -5 & 10 & -10 & 5 \\ 0 & 1 & -5 & 10 & -10 \\ 0 & 0 & 1 & -5 & 10 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.1}$$

[17] The above results imply conjecture $2a$ in the first paper and are in conflict with the conjecture 1, as the generator T_∞ defined in section 7.2 is not a large volume monodromy.

With the choice $\{R_a\} = \{\mathcal{O}(-4), \mathcal{O}(-3), \ldots, \mathcal{O}\}$ we obtain from (4.6) the following Chern characters of the duals S^a:

$$\begin{array}{c|ccccc} & r & \mathrm{ch}_1\,[K^1] & \mathrm{ch}_2\,[K^2] & \mathrm{ch}_3\,[K^3] & \mathrm{ch}_4\,[K^4] \\ \hline S^1 & 1 & -1 & 1/2 & -1/6 & 1/24 \\ -S^2 & 4 & -3 & 1/2 & 1/2 & -11/24 \\ S^3 & 6 & -3 & -1/2 & 1/2 & 11/24 \\ -S^4 & 4 & -1 & -1/2 & -1/6 & -1/24 \\ S^5 & 1 & 0 & 0 & 0 & 0 \end{array} \qquad \text{(A.2)}$$

Upon restriction to the quintic Calabi-Yau manifold Y described by the quintic $p_Y = \sum_{i=1}^5 x_i^5 = 0$, these expressions agree with the result in [8], where the fractional branes have been identified using the cumbersome analytic continuation of period matrices performed in ref. [51]. The above Chern characters agree with that of the bundles[18]

$$S^a = (-1)^{a-1} \Lambda^{a-1} \Omega^*(-a), \;\; a = 1, \ldots, 5\,. \qquad \text{(A.3)}$$

B. Fractional branes on the manifold $\mathbb{WP}^4_{1,1,1,6,9}[18]$

Here we give another instructive example for the structure of the fractional branes. It is the case of the degree 18 hypersurface $\mathbb{WP}^4_{1,1,1,6,9}[18]$ with $h_{1,1} = 1$ considered in [32], based on the earlier study of [54]. We want to describe the fractional branes in terms of tangent bundles on submanifolds of Y. The manifold Y is a simple elliptic fibration over a $\mathbb{P}^2$, which we denote by E. Due to the fibration structure, the fermionic zero modes associated with E describe literally the tangent bundle on $\mathbb{P}^2$. Let $i : E \to Y$ denote again the inclusion, $\pi : X \to E$ the fibration and $\mathcal{L}(a, b)$ the twist of the bundle $\mathcal{L}$ by the line bundle with Chern classes $a\,K_1 + b\,K_2$. The 18 fractional branes V^a come in a set of 9 branes and their anti-branes. To be precise, this is only true for the restrictions $V^a = S^a|_Y$, but not for the branes on the total space. We take, as before, a foundation $\{R_a\}$ left to the origin, that is $R_{18} = \mathcal{O}$. Then the first nine fractional branes are the sheaves $V^a = \mathcal{O}(-1, 2) \otimes \tilde{V}^a$ with

$$\begin{array}{lll} \tilde{V}^1 = +\mathcal{O}(0,0)\,, & \tilde{V}^4 = -i_*\,\mathcal{O}(0,-3)\,, & \tilde{V}^7 = -\mathcal{O}(1,-3)\,, \\ \tilde{V}^2 = -\pi^*\Omega^*_{\mathbb{P}^2}(0,-1)\,, & \tilde{V}^5 = +i_*\,\Omega^*_{\mathbb{P}^2}(0,-4)\,, & \tilde{V}^8 = +\pi^*\Omega^*_{\mathbb{P}^2}(1,-4), \\ \tilde{V}^3 = +\mathcal{O}(0,1)\,, & \tilde{V}^6 = -i_*\,\mathcal{O}(0,-2)\,, & \tilde{V}^9 = -\mathcal{O}(1,-2)\,. \end{array} \qquad \text{(B.1)}$$

We see that the fractional branes come in blocks of three which originate from the foundation $\mathcal{E}_S = \{\mathcal{O}, \Omega^*(-1), \wedge^2\Omega^*(-2){=}\mathcal{O}(1)\}$ of the helix on $\mathbb{P}^2$. The first block $\{V^1, V^2, V^3\}$ is the pull back of $\mathcal{E}_S$ to Y, the second one, $\{V^4, V^5, V^6\}$, is the extension of $\mathcal{E}_S \otimes \mathcal{O}(0, -3)$ by zero and the third is again a pull back twisted by $\mathcal{O}(1, -3)$. The twist bundles are easily recognized as the charges of the matter field z which describes the embedding of E as a divisor in Y, $E : z{=}0$. They describe the normal bundle of E in Y.

[18]The following agreement was independently noted by A. Lutken.

The pattern of the repetition of the exceptional collection $\mathcal{E}_S$ is related to the shift of the B-field by K_1. In fact the mutation $\underline{R}^{N-1}R_1$ that shifts the origin by one to the right is the monodromy

$$T_\infty^{-1} = \begin{pmatrix} 1 & -1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 1 \\ 0 & 3 & -9 & 1 & 0 & -2 \end{pmatrix}, \tag{B.2}$$

which may be again written as $T_\infty = AT$, where T is the monodromy of a massless D6-brane as in (9.15), and A the monodromy at the Gepner point. In particular the monodromy T_∞^3 for the shift of the origin of $\{R_a\}$ by three units preserves the large volume limit and coincides with the shift of the B-field, $t_1 \to t_1 + 1$.

C. A quotient singularity without crepant resolution

A simple example where the LSM resolution does not provide a complete crepant resolution is the quotient singularity $Z = \mathbb{C}^5/\mathbb{Z}_6$ specified by the weight vector $w = (1,1,1,1,2)$. In this case the index (4.3) is not well-defined and we work instead directly on the degree six hypersurface Y embedded in the exceptional divisor $X = \mathbb{P}^4_{1,1,1,1,2}$. The index on the hypersurface is given by eqs. (4.4),(4.5) and (8.3) and takes the following form on the bases $\{R_a\}$ and $\{S^a\}$:

$$I_{LG\,ab} = \begin{pmatrix} 0 & 4 & 11 & 24 & 46 & 80 \\ -4 & 0 & 4 & 11 & 24 & 46 \\ -11 & -4 & 0 & 4 & 11 & 24 \\ -24 & -11 & -4 & 0 & 4 & 11 \\ -46 & -24 & -11 & -4 & 0 & 4 \\ -80 & -46 & -24 & -11 & -4 & 0 \end{pmatrix},$$

$$I_{LG}^{ab} = \begin{pmatrix} 0 & -4 & 5 & 0 & -5 & 4 \\ 4 & 0 & -4 & 5 & 0 & -5 \\ -5 & 4 & 0 & -4 & 5 & 0 \\ 0 & -5 & 4 & 0 & -4 & 5 \\ 5 & 0 & -5 & 4 & 0 & -4 \\ -4 & 5 & 0 & -5 & 4 & 0 \end{pmatrix}. \tag{C.1}$$

The generators R_a in the large volume limit of Y coincides with the definition in the LG phase, $\{R_a\} = \{\mathcal{O}(-5), \ldots, \mathcal{O}\}$. Note that the definition of the basis S^a, as described in sections 5.4 and 7.5, is in terms of sequences on Y, as the original sequences on the singular X will not be exact.

References

[1] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B 438** (1995) 109 [hep-th/9410167];
E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124];
A. Strominger, *Massless black holes and conifolds in string theory*, *Nucl. Phys.* **B 451** (1995) 96 [hep-th/9504090];
J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017].

[2] M. Kontsevich, *Homological algebra of mirror symmetry*, alg-geom/9411018

[3] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N=2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[4] C. Vafa, *Extending mirror conjecture to Calabi-Yau with bundles*, [hep-th/9804131].

[5] D.R. Morrison, *The geometry underlying mirror symmetry*, alg-geom/9608006.

[6] H. Ooguri, Y. Oz and Z. Yin, *D-branes on Calabi-Yau spaces and their mirrors*, *Nucl. Phys.* **B 477** (1996) 407 [hep-th/9606112].

[7] A. Recknagel and V. Schomerus, *D-branes in gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186].

[8] I. Brunner, M.R. Douglas, A. Lawrence and C. Romelsberger, *D-branes on the quintic*, *J. High Energy Phys.* **08** (2000) 015 [hep-th/9906200].

[9] M.R. Douglas and G. Moore, *D-branes, quivers and ale instantons*, [hep-th/9603167].

[10] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[11] M.R. Douglas, *D-branes and discrete torsion*, hep-th/9807235;
M.R. Douglas and B. Fiol, *D-branes and discrete torsion. II*, hep-th/9903031.

[12] M.R. Douglas, B. Fiol and C. Romelsberger, *The spectrum of BPS branes on a noncompact Calabi-Yau*, hep-th/0003263;*Stability and BPS branes*, hep-th/0002037.

[13] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.

[14] D.-E. Diaconescu and M.R. Douglas, *D-branes on stringy Calabi-Yau manifolds*, hep-th/0006224.

[15] M.R. Douglas, *D-branes on Calabi-Yau manifolds*, math.AG/0009209

[16] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[17] For a formulation in terms of toric geometry, see: D.R. Morrison and M.R. Plesser, *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, *Nucl. Phys.* **B 440** (1995) 279 [hep-th/9412236].

[18] E. Witten, *Constraints on supersymmetry breaking*, *Nucl. Phys.* **B 202** (1982) 253.

[19] R. Rohm and E. Witten, *The antisymmetric tensor field in superstring theory*, *Ann. Phys. (NY)* **170** (1986) 454.

[20] W. Lerche, C. Vafa and N.P. Warner, *Chiral rings in N=2 superconformal theories*, *Nucl. Phys.* **B 324** (1989) 427.

[21] A.N. Rudakov ed., *Helices and vector bundles: seminaire Rudakov*, *London Mathematical Society Lecture Note Series 148*, Cambridge University Press, Cambridge 1990.

[22] E. Zaslow, *Solitons and helices: the search for a math physics bridge*, *Comm. Math. Phys.* **175** (1996) 337 [hep-th/9408133].

[23] Y. Ito and H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, math.AG/9803120

[24] K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222.

[25] M. Gutperle and Y. Satoh, *D-branes in gepner models and supersymmetry*, *Nucl. Phys.* **B 543** (1999) 73 [hep-th/9808080];
S. Govindarajan and T. Jayaraman, *On the Landau-Ginzburg description of boundary cfts and special lagrangian submanifolds*, *J. High Energy Phys.* **07** (2000) 016 [hep-th/0003242];
I. Brunner and V. Schomerus, *On superpotentials for D-branes in gepner models*, *J. High Energy Phys.* **10** (2000) 016 [hep-th/0008194].

[26] S. Govindarajan, T. Jayaraman and T. Sarkar, *On D-branes from gauged linear sigma models*, *Nucl. Phys.* **B 593** (2001) 155 [hep-th/0007075].

[27] N.P. Warner, *Supersymmetry in boundary integrable models*, *Nucl. Phys.* **B 450** (1995) 663 [hep-th/9506064].

[28] S. Govindarajan, T. Jayaraman and T. Sarkar, *Worldsheet approaches to D-branes on supersymmetric cycles*, *Nucl. Phys.* **B 580** (2000) 519 [hep-th/9907131].

[29] Second ref. in [33]

[30] J.A. Harvey and G. Moore, *On the algebras of BPS states*, *Comm. Math. Phys.* **197** (1998) 489 [hep-th/9609017].

[31] E. Witten, *D-branes and k-theory*, *J. High Energy Phys.* **12** (1998) 019 [hep-th/9810188].

[32] D.-E. Diaconescu and C. Romelsberger, *D-branes and bundles on elliptic fibrations*, *Nucl. Phys.* **B 574** (2000) 245 [hep-th/9910172].

[33] M.B. Green, J.A. Harvey and G. Moore, *I-brane inflow and anomalous couplings on D-branes*, *Class. and Quant. Grav.* **14** (1997) 47 [hep-th/9605033];
R. Minasian and G. Moore, *K-theory and Ramond-Ramond charge*, *J. High Energy Phys.* **11** (1997) 002 [hep-th/9710230];
Y.-K.E. Cheung and Z. Yin, *Anomalies, branes and currents*, *Nucl. Phys.* **B 517** (1998) 69 [hep-th/9710206].

[34] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, hep-th/9907189.

[35] W. Lerche and N.P. Warner, *Index theorems in $N=2$ superconformal theories*, *Phys. Lett.* **B 205** (1988) 471.

[36] S. Cecotti and C. Vafa, *Topological antitopological fusion*, *Nucl. Phys.* **B 367** (1991) 359.

[37] S. Cecotti and C. Vafa, *On classification of $N = 2$ supersymmetric theories*, *Comm. Math. Phys.* **158** (1993) 569 [hep-th/9211097].

[38] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, *Journ. Alg. Geom.* **3** (1994) 493; *Duke Math. Journ.* **69** (1993) 349.

[39] V.V. Batyrev and D.I. Dais, *Strong McKay correspondence, string theoretic Hodge numbers and mirror symmetry*, alg-geom/9410001

[40] M. Kontsevich, as quoted in [13]

[41] A.I. Bondal,*Helices, representations of quivers and Koszul algebras*, in [21], *Math. USSR Izv.* **34** (1990) 23.

[42] M. Douglas, private communication.

[43] D.-E. Diaconescu and J. Gomis, *Fractional branes and boundary states in orbifold theories*, *J. High Energy Phys.* **10** (2000) 001 [hep-th/9906242].

[44] P. Kaste, W. Lerche, C.A. Lutken and J. Walcher, *D-branes on k3-fibrations*, *Nucl. Phys.* **B 582** (2000) 203 [hep-th/9912147];
J. Fuchs et al., *Boundary fixed points, enhanced gauge symmetry and singular bundles on k3*, [hep-th/0007145];
W. Lerche, C.A. Lutken and C. Schweigert, *D-branes on ale spaces and the ade classification of conformal field theories*, [hep-th/0006247];
W. Lerche, *On a boundary CFT description of nonperturbative $N = 2$ Yang-Mills theory*, [hep-th/0006100].

[45] E. Scheidegger, *D-branes on some one- and two-parameter Calabi-Yau hypersurfaces*, *J. High Energy Phys.* **04** (2000) 003 [hep-th/9912188].

[46] C. Beasley, B.R. Greene, C.I. Lazaroiu and M.R. Plesser, *D3-branes on partial resolutions of abelian quotient singularities of Calabi-Yau threefolds*, *Nucl. Phys.* **B 566** (2000) 599 [hep-th/9907186];
B. Feng, A. Hanany and Y.-H. He, *D-brane gauge theories from toric singularities and toric duality*, hep-th/0003085;
P.S. Aspinwall and M.R. Plesser, *D-branes, discrete torsion and the mckay correspondence*, hep-th/0009042.

[47] A. Ceresole, R. D'Auria, S. Ferrara and A.V. Proeyen, *Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity*, *Nucl. Phys.* **B 444** (1995) 92 [hep-th/9502072].

[48] *Essays on mirror manifolds*, S.T. Yau ed., Int. Press, Hong Kong 1992.

[49] S. Hosono, A. Klemm, S. Theisen and S.T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, *Nucl. Phys.* **B 433** (1995) 501 [hep-th/9406055].

[50] S. Hosono, *Local mirror symmetry and type-IIA monodromy of Calabi-Yau manifolds*, hep-th/0007071.

[51] P. Candelas, X.C. De La Ossa, P.S. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys.* **B 359** (1991) 21.

[52] A.C. Avram, E. Derrick and D. Jancic, *On semi-periods*, *Nucl. Phys.* **B 471** (1996) 293 [hep-th/9511152].

[53] S. Govindarajan and T. Jayaraman, *D-branes, exceptional sheaves and quivers on Calabi-Yau manifolds: from Mukai to Mckay*, hep-th/0010196;
A. Tomasiello, *D-branes on Calabi-Yau manifolds and helices*, hep-th/0010217.

[54] P. Candelas, A. Font, S. Katz and D.R. Morrison, *Mirror symmetry for two parameter models. 2*, *Nucl. Phys.* **B 429** (1994) 626 [hep-th/9403187].